

TUTTE POLYNOMIALS FOR DIRECTED GRAPHS

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ABSTRACT. The Tutte polynomial is a fundamental invariant of graphs. In this article, we define and study a generalization of the Tutte polynomial for directed graphs, that we name *B-polynomial*. The *B-polynomial* has three variables, but when specialized to the case of graphs (that is, digraphs where arcs come in pairs with opposite directions), one of the variables becomes redundant and the *B-polynomial* is equivalent to the Tutte polynomial. We explore various properties, expansions, specializations, and generalizations of the *B-polynomial*, and try to answer the following questions:

- what properties of the digraph can be detected from its *B-polynomial* (acyclicity, length of paths, number of strongly connected components, etc.)?
- which of the marvelous properties of the Tutte polynomial carry over to the directed graph setting?

The *B-polynomial* generalizes the strict chromatic polynomial of mixed graphs introduced by Beck, Bogart and Pham. We also consider a quasisymmetric function version of the *B-polynomial* which simultaneously generalizes the Tutte symmetric function of Stanley and the quasisymmetric chromatic function of Shareshian and Wachs.

1. INTRODUCTION

The Tutte polynomial is a fundamental invariant of graphs. There is a vast and rich literature about the Tutte polynomial; see for instance [11, 34] or [10, Chapter 10] for an introduction. In this article, we investigate a generalization of the Tutte polynomial to all directed graphs (or *digraphs* for short). Note that graphs are a special case of digraphs: they are the digraphs such that arcs come in pairs with opposite directions.

Several digraph analogues of the Tutte polynomials have been considered in the literature. This includes in particular the *cover polynomial* of Chung and Graham [13], and the *Gordon-Traldi polynomials* [17]. See [12] and references therein for an overview of these digraph invariants. These digraph invariants do share some of the features of the Tutte polynomial. However most are not proper generalizations of the Tutte polynomial, as they are not equivalent to the Tutte polynomial for the special case of graphs. The only exception is the Gordon-Traldi polynomial denoted f_8 in [17], which is an invariant for *vertex-ordered digraphs* (pairs made of a digraph and a linear order of its vertices).

In the present article, we define a new digraph invariant $B_D(q, y, z)$, which is a polynomial in three variables generalizing the Tutte polynomial to all digraphs D . Precisely, when the digraph D corresponds to a graph, then $B_D(q, y, z)$ is *equivalent* (that is, equal up to a change of variables), to the Tutte polynomial $T_D(x, y)$. In effect, the third variable z becomes redundant in the case of graphs, but it is not for general digraphs. There are actually two additional relations between our digraph invariant and the Tutte polynomial. First, for any graph G , $T_G(x, y)$ is equivalent to the average of $B_D(q, y, z)$ over all digraphs obtained by orienting G . Second, for any digraph D , $B_D(q, y, y)$ is equivalent to the Tutte polynomial $T_G(x, y)$ of the *graph G underlying D* (that is, the graph obtained by forgetting the direction of the arcs). All these properties make the *B-polynomial* a very legitimate generalization of the Tutte polynomial.

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Without further ado, let us define the B -polynomial. For a digraph $D = (V, A)$, we define $B_D(q, y, z)$ as the unique polynomial in the variables q, y, z such that for any positive integer q ,

$$B_D(q, y, z) := \sum_{f: V \rightarrow \{1, 2, \dots, q\}} y^{\#\{(u, v) \in A, f(v) > f(u)\}} z^{\#\{(u, v) \in A, f(v) < f(u)\}}. \quad (1)$$

In words, the B -polynomial counts the q -colorings of D (arbitrary functions from V to $\{1, 2, \dots, q\}$) according to the number of strict ascents and strict descents. As we show in Section 3, the three relations stated above between the B -polynomial and the Tutte polynomial follow pretty easily from the known equivalence between the Tutte polynomial of a graph G and the *partition function of the Potts model on G* (or *Potts polynomial* for short).

Example 1.1. For the digraphs represented in Figure 1, one gets

$$\begin{aligned} B_D(q, y, z) &= q + \frac{q(q-1)}{2}(y+z), \\ B_{D'}(q, y, z) &= q + q(q-1)(y^2 + z^2 + yz) + \frac{q(q-1)(q-2)}{6}(y^3 + z^3 + 2yz(y+z)), \\ B_{D''}(q, y, z) &= q + q(q-1)yz(y+z+1) + \frac{q(q-1)(q-2)}{6}yz(y^2 + z^2 + 4yz). \end{aligned}$$

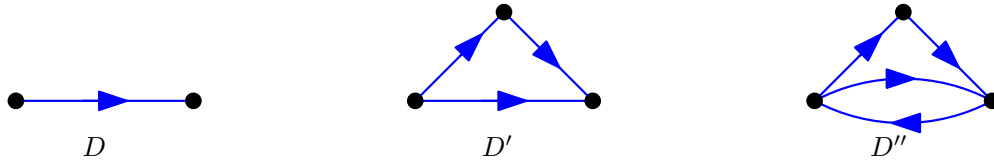


FIGURE 1. Three digraphs.

Our goal is to investigate how much of the theory of the Tutte polynomial extends to the digraph setting, and unearth some new identities. Some highlights are the following.

- The invariant B_D detects whether D is acyclic. More generally, B_D contains the generating function of acyclic subgraphs of D , counted according to their number of arcs. This generalizes to digraphs the result of Stanley about acyclic orientations of graphs [29], and also the result of Backman and Hopkins about acyclic fourorientations of graphs [4, 5]. Moreover, B_D contains the generating function of acyclic reorientations of D , counted according to their number of reoriented arcs (equivalently, according to their distance to D in the graph of reorientations). This refines the result of Stanley about the acyclic orientations of the underlying graph.
- The invariant B_D detects whether D is totally cyclic. More generally, B_D contains the generating function of totally cyclic contractions of D , counted according to their number of arcs. Moreover, B_D contains the generating function of totally cyclic reorientations of D , counted according to their number of reoriented arcs.
- Digraphs can also be seen as *mixed graphs* (a.k.a. *partially oriented graphs*) by interpreting the pairs of arcs in opposite direction as *unoriented edges*. With this perspective, the invariant B_D contains the number of ways of orienting the unoriented edges to get an acyclic orientation. This result was already obtained by Beck, Bogard, and Pham in [7], who defined and studied a specialization of B_D which they call *strict Chromatic polynomial*. We also obtain a generalization to mixed graphs of a result of Las Vergnas about the number of totally cyclic orientations of a graph [19].

- The invariant B_D detects the length of the longest directed path in D , the number of strongly-connected components, and the number of linear extensions of the acyclic graph obtained from D by contracting all the strongly-connected components.
- When D is planar, there is a simple relation between $B_D(-1, y, z)$ and $B_{D^*}(-1, y, z)$, where D^* is a dual digraph.
- When D is acyclic, there is a simple relation between $B_D(-q, y, 1)$ and $B_D(q, y, 1)$, and when D is a tree, there is simple relation between $B_D(-q, y, z)$ and $B_D(q, y, z)$.
- B_D has a *quasi-symmetric function* generalization $B_D(\mathbf{x}; y, z)$. The invariant $B_D(\mathbf{x}; y, z)$ generalizes to digraphs the *Tutte symmetric function* defined by Stanley for graphs [31], and the chromatic quasisymmetric function defined by Shareshian and Wachs for acyclic digraphs [24].

Actually, the B -polynomial is not the only natural generalization of the Tutte polynomial to digraphs. Indeed, in Section 9, we present an infinite family of polynomial invariants of digraphs which generalize the B -polynomials and satisfy the same three stated connections with the Tutte polynomial. The B -polynomial is merely the “simplest” of these invariants. In the companion paper [3], we investigate another of these invariants from this infinite family that we call *A-polynomial*. Unlike the B -polynomial, the A -polynomial is an *oriented matroid invariant*, that is, it only depends on the oriented matroid which underlies the digraph.

The paper is organized as follows. In Section 2, we set up notation and definitions about digraphs. In Section 3, we define the B -polynomial and relate it to the Potts polynomial and Tutte polynomial. In Section 4, we explore the consequences of recurrence relations for the B -polynomial. In Section 5, we give several expansions of the B -polynomial in terms of the oriented chromatic polynomials. In Section 6, we use Ehrhart theory to give an interpretation of the B -polynomial at negative values of q . In Section 7, we investigate several evaluations of the B -polynomials. In Section 8, we present a quasisymmetric function generalization of the B -polynomial, and use the theory of P -partitions to prove some new properties. In Section 9, we present a family of invariants generalizing the B -polynomial. We conclude in Section 10 with a list of open problems.

2. NOTATION AND DEFINITIONS ABOUT GRAPHS AND DIGRAPHS

In this section we set some basic notation. We denote \mathbb{N} the set of non-negative integers, and \mathbb{P} the set of positive integers. For a positive integer n , we denote $[n] = \{1, \dots, n\}$. We denote \mathfrak{S}_n the set of permutations of $[n]$, and for integers $a < b$, we denote $[a..b] := \{a, a+1, \dots, b\}$. For a set S , we denote $|S|$ its cardinality. For sets R, S, T we denote $R \uplus S = T$ to indicate that T is the *disjoint union* of R and S . For a polynomial P in a variable x , we denote $[x^k]P$ the coefficient of x^k in P . We denote $\deg_x(P)$ the *degree* of the polynomial P in the variable x . For a condition C , the symbol $\mathbf{1}_C$ has value 1 if the condition C is true and 0 otherwise.

A *graph* is a pair $G = (V, E)$, where V is a finite set of *vertices* and E is a finite set of *edges* which are subsets $\{u, v\}$ of 1 or 2 vertices (we accept $u = v$). We denote by $v(G)$ the number of vertices of G , and by $c(G)$ the number of connected components of G .

A *directed graph*, or *digraph* for short, is a pair (V, A) , where V is a finite set of *vertices* and A is a finite set of *arcs* which are pairs (u, v) of vertices. We authorize our digraphs to have *loops* (that is, arcs of the form (u, u)) and *multiple arcs* (so that A is really a *multiset* which can contain multiple copies of the element (u, v)). We say that the arc $a = (u, v)$ has *origin* u , *target* v , and *endpoints* u and v . We say that the arc (v, u) is the *opposite* of the arc (u, v) .

For a digraph $D = (V, A)$, we call *underlying graph* the graph denoted \underline{D} , obtained by replacing each arc (u, v) by the edge $\{u, v\}$. Conversely, an *orientation* of a graph G is a digraph D with underlying graph G : it is obtained by choosing a direction for each edge $\{u, v\}$. For a graph $G = (V, E)$, we denote $\vec{G} = (V, A)$ the digraph obtained by replacing each edge $\{u, v\} \in E$ by the two opposite arcs (u, v) and (v, u) . This operation identifies the set of graphs with the subset of

digraphs such that for all $u, v \in V$ the arc (u, v) appears with the same multiplicity as the opposite arc (v, u) .

Let $D = (V, A)$ be a digraph. A set of arcs $S \subseteq A$ is a *cocycle* if there exists a partition of the vertex set $V = V_1 \uplus V_2$ such that S is the set of edges with one endpoint in V_1 and one endpoint in V_2 . It is a *directed cocycle* if moreover the origins of the arcs in S are all in the same subset of vertices, say V_1 . An arc $a \in A$ is called *cyclic* if it is in a directed cycle, and *acyclic* if it is in a directed cocycle. As is well known, any arc is either cyclic or acyclic, but never both. The digraph D is *acyclic* if every arc is acyclic (i.e. D has no directed cycles), and *totally cyclic* if every arc is cyclic (i.e. D has no directed cocycle).

We now define the *deletion*, *contraction*, and *reorientation* of digraphs. These operations are represented in Figure 2. Let $D = (V, A)$ be a digraph, and let $a = (u, v) \in A$ be an arc.

- We denote $D \setminus a$ the digraph obtained by *deleting* a , that is, removing a from A .
- We denote $D_{/a}$ the digraph obtained by *contracting* a , that is, removing a from A and identifying its two endpoints u and v . Note that if $u = v$, then $D_{/a} = D \setminus a$.
- We denote D^{-a} the digraph obtained by *reorienting* a , that is, replacing $a = (u, v)$ by the opposite arc $-a = (v, u)$.

Note that these three operations commute with each other. For instance, for any arc $a, b \in A$, $D_{/a/b} = D_{/b/a}$ and $D \setminus a/b = D_{/b \setminus a}$. Hence, the definition of deletion, contraction and reorientation can be extended to sets of arcs: for any disjoint set of arcs $R, S, T \subseteq A$ we denote $D_{\setminus R/S}^{-T}$ the digraph obtained from D by deleting the arcs in R , contracting the arcs in S , and reorienting the arcs in T .

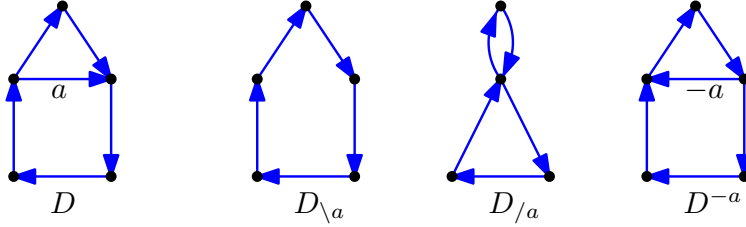


FIGURE 2. The result of deleting, contracting, and reorienting an arc a of D .

3. THE B -POLYNOMIAL AND ITS RELATION TO THE POTTS AND TUTTE POLYNOMIALS

In this section we define the B -polynomial of digraphs, and study its most immediate properties. In particular, we establish the existence of a polynomial B_D satisfying (1), and establish three relations between B_D and the Tutte polynomial.

Let $D = (V, A)$ be a digraph, and let q be a positive integer. We call q -coloring of D a function from V to $[q]$, and we call $f(v)$ the *color* of the vertex v . For any function from V to \mathbb{Z} we denote $f_A^>$ the set of arcs $(u, v) \in A$ such that $f(v) > f(u)$. We define the set of arcs $f_A^<$, f_A^{\geq} , f_A^{\leq} , $f_A^=$, f_A^{\neq} similarly. We call the elements in $f_A^>$ and $f_A^<$ the *ascents* and *descents* of f respectively. A q -coloring f is *proper* if $f_A^= = \emptyset$.

Theorem 3.1. *Let $D = (V, A)$ be a digraph. There exists a (unique) trivariate polynomial $B_D(q, y, z)$ such that for all positive integers q ,*

$$B_D(q, y, z) = \sum_{f: V \rightarrow [q]} y^{|f_A^>|} z^{|f_A^<|}.$$

where the sum is over all possible q -colorings of D . We call $B_D(q, y, z)$ the B -polynomial of D .

Proof. The proof of Theorem 3.1 is based on the reduction from general q -colorings to *surjective* q -colorings (surjective functions from V to $[q]$). We denote $\text{Surj}(V, q)$ the set of surjective q -colorings. Given an arbitrary q -coloring f , we consider the number of colors used $p = |f(V)| \leq |V|$, and the unique order-preserving bijection φ from $f(V)$ to $[p]$. We denote \tilde{f} the surjective p -coloring $\varphi \circ f$. Observe that $|f_A^>| = |\tilde{f}_A^>|$ and $|f_A^<| = |\tilde{f}_A^<|$. Hence,

$$\begin{aligned} \sum_{f:V \rightarrow [q]} y^{|f_A^>|} z^{|f_A^<|} &= \sum_{p=1}^{|V|} \sum_{g \in \text{Surj}(V, p)} \sum_{f:V \rightarrow [q] \text{ such that } \tilde{f}=g} y^{|f_A^>|} z^{|f_A^<|}, \\ &= \sum_{p=1}^{|V|} \sum_{g \in \text{Surj}(V, p)} y^{|g_A^>|} z^{|g_A^<|} \left| \{f:V \rightarrow [q] \text{ such that } \tilde{f}=g\} \right|, \\ &= \sum_{p=1}^{|V|} \binom{q}{p} \sum_{g \in \text{Surj}(V, p)} y^{|g_A^>|} z^{|g_A^<|}, \end{aligned}$$

where the third line uses the observation that a q -coloring f is uniquely identified by the surjective p -coloring \tilde{f} and the set $f(V)$ (there are $\binom{q}{p}$ ways to choose this set). Thus, the trivariate polynomial

$$B_D(q, y, z) = \sum_{p=1}^{|V|} \frac{q(q-1) \dots (q-p+1)}{p!} \sum_{g \in \text{Surj}(V, p)} y^{|g_A^>|} z^{|g_A^<|}, \quad (2)$$

satisfies the properties of the theorem. The uniqueness of B_D is obvious since distinct polynomials in q cannot agree on infinitely many values of q . \square

We now state some immediate properties of the B -polynomial.

Proposition 3.2. *For any digraph $D = (V, A)$, the B -polynomial has the following properties.*

- (a) $B_D(q, y, z) = B_D(q, z, y)$.
- (b) *Reorienting all the arcs of D does not change the B -polynomial:* $B_{D-A}(q, y, z) = B_D(q, y, z)$.
- (c) $\sum_{f:V \rightarrow [q]} x^{|f_A^=|} y^{|f_A^>|} z^{|f_A^<|} = x^{|A|} B_D\left(q, \frac{y}{x}, \frac{z}{x}\right)$.
- (d) $\sum_{f:V \rightarrow [q]} y^{|f_A^>|} z^{|f_A^<|} = (yz)^{|A|} B_D\left(q, \frac{1}{y}, \frac{1}{z}\right)$.
- (e) *The B -polynomial of the digraph with a single vertex and no arcs is q .*
- (f) *If $a = (u, u) \in A$ is a loop, then $B_{D \setminus a}(q, y, z) = B_D(q, y, z)$.*
- (g) *If D is the disjoint union of two digraphs D_1 and D_2 , then $B_D(q, y, z) = B_{D_1}(q, y, z) B_{D_2}(q, y, z)$.*
- (h) $\deg_q(B_D(q, y, z)) = |V|$ and $B_D(q, 1, 1) = q^{|V|}$. *If D has no loops, $\deg_y(B_D(q, y, y)) = |A|$.*
- (i) *The polynomial $B_D(q, y, z)$ is divisible by $q^{c(D)}$, and $B_D(q, 0, 0) = q^{c(D)}$.*
- (j) *The expansion of the polynomial $B_D(q, y, z)$ in the basis $\{\frac{q(q-1) \dots (q-p+1)}{p!} y^i z^j\}_{p>0, i, j \geq 0}$ has positive integer coefficients.*
- (k) $|V|! [q^{|V|}] B_D(q, y, z) = \sum_{\text{bijection } f:V \rightarrow [|V|]} y^{|f_A^>|} z^{|f_A^<|}$.
- (l) $B_D(2, 1, 0)$ *is the number of directed cuts of G (that is, the number of subsets of vertices $U \subset V$ such that every arc joining U to $V \setminus U$ is oriented away from U).*

Proof. Property (a) is easy to prove using the involution on q -colorings which changes the color $i \in [q]$ by $q+1-i$ (so that ascents become descents and vice-versa). The same reasoning gives (b). Property (c) is clear from the fact that $|f_A^=| = |A| - |f_A^>| - |f_A^<|$. Property (d) is clear from (a) and (c). It shows that the B -polynomial can be equivalently thought as counting q -colorings according to

the number of *weak ascents* and *weak descents*. The properties (e), (f), (g) are obvious from the definitions. For (h), observe that $B_D(q, 1, 1)$ counts all q -colorings, hence $B_D(q, 1, 1) = q^{|V|}$. Moreover $\deg_q(B_D(q, y, z))$ cannot be more than $|V|$ by (2). Clearly $\deg_y(B_D(q, y, y)) \leq |A|$, and considering proper q -colorings gives equality for loopless digraphs. For (i), observe that (2) shows that q divides $B_D(q, y, z)$. Hence, by (g), $q^{c(D)}$ divides $B_D(q, y, z)$. Moreover, $B_D(q, 0, 0)$ counts q -colorings which are constant on each connected components, and there are $q^{c(D)}$ such q -colorings. Property (j) and (k) follow directly from (2). Lastly, (l) is clear from the definitions upon identifying each 2-coloring f with the subset of vertices $U_f = \{v \in V \mid f(v) = 1\}$. \square

Remark 3.3. We will see that a number of properties of a digraph D (acyclicity, maximal length of directed paths, etc.) can be read off the invariant $B_D(q, y, z)$. However, Property (b) of Proposition 3.2 shows that certain properties cannot be read off $B_D(q, y, z)$. In particular, the outdegree distribution of D , the number of sources of D , or the number of *directed spanning trees* (spanning trees such that every vertex except one has indegree 1) cannot be read off $B_D(q, y, z)$.

Example 3.4. Let us illustrate the significance of Proposition 3.2(k) for the digraphs D , D' , D'' represented in Figure 3. Note that, up to assuming $V = [n]$ for some integer n , the sum in (k) is over the permutations of $[n]$; and these permutations are counted according to some statistics described by A . For instance, if D is a directed path then the statistics are the number of ascents and descents. More precisely, if $D = ([n], A)$ is the directed path with arc set $A = \{(i, i+1) \mid i \in [n-1]\}$, then (k) gives

$$n![q^n]B_D(q, y, z) = \sum_{\sigma \in \mathfrak{S}_n} y^{\text{asc}(\sigma)} z^{\text{des}(\sigma)}, \quad (3)$$

where $\text{asc}(\sigma)$ and $\text{des}(\sigma)$ are the number of *ascents* ($i \in [n-1]$ such that $\sigma(i) < \sigma(i+1)$) and *descents* ($i \in [n-1]$ such that $\sigma(i) > \sigma(i+1)$) of the permutation σ .

If $D' = ([n], A)$ is the digraph with i copies of the arc $(i, i+1)$ for all $i \in [n-1]$, then (k) gives

$$n![q^n]B_{D'}(q, y, z) = \sum_{\sigma \in \mathfrak{S}_n} y^{\binom{n}{2} - \text{maj}(\sigma)} z^{\text{maj}(\sigma)}, \quad (4)$$

where $\text{maj}(\sigma) = \sum_{\substack{i \in [n-1] \\ \sigma(i) > \sigma(i+1)}} i$ is the *major index*. By a classical formula (see [33, Prop 1.4.6]), the

right-hand side of (4) is $\prod_{k=1}^n \left(\sum_{i=1}^k y^{i-1} z^{k-i} \right)$.

Lastly, if $D'' = ([n], \{(u, v) \mid 1 \leq u < v \leq n\})$, then (k) gives

$$n![q^n]B_{D''}(q, y, z) = \sum_{\sigma \in \mathfrak{S}_n} y^{\binom{n}{2} - \text{inv}(\sigma)} z^{\text{inv}(\sigma)}, \quad (5)$$

where $\text{inv}(\sigma)$ is the number of *inversions* (pairs $(i, j) \in [n]^2$ with $i < j$ and $\sigma(i) > \sigma(j)$). By a classical formula (see [33, Cor 1.3.13]), the right-hand side of (5) is again $\prod_{k=1}^n \left(\sum_{i=1}^k y^{i-1} z^{k-i} \right)$. These formulas will be refined in Section 8 (see Example 8.16).

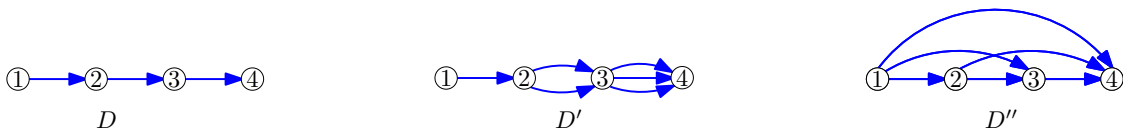


FIGURE 3. Digraphs giving the descent, major index, and inversion statistics of permutations.

Next, we prove the relations stated in the introduction between the B -polynomial and the Tutte polynomial. Recall that the *Tutte polynomial* of a graph $G = (V, E)$ is

$$T_G(x, y) = \sum_{S \subseteq E} (x-1)^{c(S)-c(E)} (y-1)^{|S|+c(S)-|V|}, \quad (6)$$

where the sum is over all subsets S of edges, and $c(S)$ is the number of connected components of the subgraph (V, S) . We refer the reader to [34] or [10, Chapter 10] for an introduction to the theory of the Tutte polynomial. As shown by Fortuin and Kasteleyn [14], the Tutte polynomial is equivalent to the *partition function of the Potts model on G* , or *Potts polynomial of G* for short. The *Potts polynomial of $G = (V, E)$* is the unique bivariate polynomial $P_G(q, y)$ such that for all positive integers q ,

$$P_G(q, y) = \sum_{f: V \rightarrow [q]} y^{|f_E^\neq|},$$

where f_E^\neq represents the set of edges with endpoints of different colors¹. Indeed, the Tutte polynomial T_G is related to P_G by the following change of variables:

$$T_G(x, y) = \frac{y^{|E|}}{(y-1)^{|V|}(x-1)^{c(G)}} P_G((x-1)(y-1), 1/y). \quad (7)$$

We now relate the B -polynomial of digraphs to the Potts polynomials of graphs.

Proposition 3.5. *Let G be a graph, and let \overleftrightarrow{G} be the corresponding digraph (obtained by replacing each edge of G by two opposite arcs). Then*

$$B_{\overleftrightarrow{G}}(q, y, z) = P_G(q, yz). \quad (8)$$

Proof. We denote $G = (V, E)$ and $\overleftrightarrow{G} = (V, A)$. We need to prove that for all positive integers q ,

$$\sum_{f: V \rightarrow [q]} y^{|f_A^>|} z^{|f_A^<|} = \sum_{f: V \rightarrow [q]} (yz)^{|f_E^\neq|}.$$

where $f_E^\neq = \{\{u, v\} \in E \mid f(u) \neq f(v)\}$. Hence, it suffices to prove that for any q -coloring f ,

$$y^{|f_A^>|} z^{|f_A^<|} = (yz)^{|f_E^\neq|}. \quad (9)$$

Consider an edge $e = \{u, v\}$ of G and the corresponding arcs $a = (u, v)$ and $b = (v, u)$ of D . If u and v have distinct colors, then one of the arcs a, b will contribute a factor of y to the left-hand side of (9), whereas the other will contribute z . Thus the arcs a, b contribute a factor yz , as does e . On the other hand, if u and v have the same color, then the arcs a, b contribute 1, as does e . \square

Proposition 3.5 shows that the B -polynomial generalizes the Potts function of graphs, or equivalently the Tutte polynomial. The second relation below shows that the Potts polynomial of a graph G is equivalent to the average of the B -polynomial over the orientations of G .

Proposition 3.6. *Let $G = (V, E)$ be a graph. Then,*

$$\frac{1}{2^{|E|}} \sum_{\vec{G} \in \text{Orient}(G)} B_{\vec{G}}(q, y, z) = P_G\left(q, \frac{y+z}{2}\right), \quad (10)$$

where $\text{Orient}(G)$ is the set of digraphs obtained by orienting G .

¹In the literature the Potts polynomial of G is more often defined as $\tilde{P}_G(q, y) = \sum_{f: V \rightarrow [q]} y^{|f_E^\neq|}$. This is equivalent to the convention in the present paper via the relation $P(q, y) = y^{|E|} \tilde{P}_G(q, 1/y)$.

Proof. For any positive integer q we can write

$$\sum_{\vec{G} \in \text{Orient}(G)} B_D(q, y, z) = \sum_{f: V \rightarrow [q]} \sum_{\vec{G} = (V, A) \in \text{Orient}(G)} y^{|f_A^>|} z^{|f_A^<|}.$$

Moreover, for any q -coloring f ,

$$\sum_{\vec{G} = (V, A) \in \text{Orient}(G)} y^{|f_A^>|} z^{|f_A^<|} = \sum_{\vec{G} = (V, A) \in \text{Orient}(G)} \prod_{(u, v) \in A} w_f(u, v) = \prod_{\{u, v\} \in E} (w_f(u, v) + w_f(v, u)),$$

where $w_f(u, v) = y$ if $f(v) > f(u)$, $w_f(u, v) = z$ if $f(v) < f(u)$, and $w_f(u, v) = 1$ if $f(v) = f(u)$. Consider an edge $e = \{u, v\}$ of G and the corresponding arcs $a = (u, v)$ and $b = (v, u)$ of D . If u and v have different colors then $w_f(u, v) + w_f(v, u) = (y + z)$, whereas if u and v have the same color then $w_f(u, v) + w_f(v, u) = 2$. Thus,

$$\sum_{\vec{G}} B_D(q, y, z) = \sum_{f: V \rightarrow [q]} (y + z)^{\#\{u, v\} \in E, f(v) \neq f(u)} 2^{\#\{u, v\} \in E, f(v) = f(u)} = 2^{|E|} P_G\left(q, \frac{y + z}{2}\right),$$

as wanted. \square

We now prove the third relation between the B -polynomial and the Potts polynomial: the B -polynomial contains the Potts polynomial of the underlying graph.

Proposition 3.7. *Let $D = (V, A)$ be a digraph, and let $\underline{D} = (V, E)$ be the underlying graph. Then,*

$$B_D(q, y, y) = P_{\underline{D}}(q, y). \quad (11)$$

Proof. For any positive integer q ,

$$B_D(q, y, y) = \sum_{f: V \rightarrow [q]} y^{|f_A^>| + |f_A^<|} = \sum_{f: V \rightarrow [q]} y^{|f_A^\neq|} = P_{\underline{D}}(q, y),$$

as wanted. \square

Remark 3.8. Proposition 3.7 implies that the polynomial $B_D(q, y, z)$ contains all the information given by the Tutte polynomial of the underlying graph \underline{D} (assuming one knows $|A|$). For instance, the number of spanning trees of \underline{D} can be read off $B_D(q, y, z)$. However, it is clear that $B_D(q, y, z)$ contains more information than $T_{\underline{D}}(x, y)$, as Proposition 3.2(1) already exemplifies.

Note that the third variable z is actually “redundant” in the relations between B -polynomial and the Potts polynomial given in Propositions 3.5, and 3.6. Indeed, for any graph $G = (V, E)$ we get

$$P_G(q, y) = B_{\overleftarrow{G}}(q, y, 1) = \frac{1}{2^{|E|}} \sum_{\vec{G}} B_{\vec{G}}(q, 2y - 1, 1). \quad (12)$$

We now introduce two variants of $B_D(q, y, 1)$ which will be convenient in order to state relations between the B -polynomial and the Tutte polynomial. For this, we adopt a different perspective on digraphs, by seeing them as *mixed graphs* (a.k.a. *partially oriented graphs*). A *mixed graph* is a graph with oriented edges and unoriented edges. Equivalently, *mixed graphs* are digraphs $D = (V, A)$ together with an *arc-partition* $E(D)$ of the arc-set A into singletons called *oriented edges*, and doubletons made of two opposite arcs called *unoriented edges*. In particular, any graph G identifies with the mixed graph \overleftrightarrow{G} without oriented edges. A *complete orientation* of a mixed graph D , is a digraph obtained by choosing a direction for each unoriented edge (i.e. replacing each doubleton in $E(D)$ by a singleton). We also denote $\text{Orient}(D)$ the set of complete orientations of D .

Definition 3.9. Let $D = (V, A)$ be a mixed graph with arc-partition $E(D)$. We define

$$T_D^{(1)}(x, y) := \frac{y^{|E(D)|}}{(y-1)^{|V|}(x-1)^{c(D)}} B_D \left((x-1)(y-1), \frac{1}{y}, 1 \right), \quad (13)$$

and

$$T_D^{(2)}(x, y) := \frac{y^{|E(D)|}}{2^{|E(D)|}(y-1)^{|V|}(x-1)^{c(D)}} \sum_{\vec{D} \in \text{Orient}(D)} B_{\vec{D}} \left((x-1)(y-1), \frac{2-y}{y}, 1 \right). \quad (14)$$

Relation (8) and (10) between the B -polynomial and the Potts function can be rewritten as follows via (7):

Corollary 3.10. For any unoriented graph G , $T_{\vec{G}}^{(1)}(x, y) = T_{\vec{G}}^{(2)}(x, y) = T_G(x, y)$.

It is clear from the definition that the invariants $T_D^{(1)}$ and $T_D^{(2)}$ are rational functions, and we will prove later that they are actually *polynomials* in x and y (see Proposition 7.1). We now show that these polynomials coincide for $y = 0$ and count *strictly-compatible colorings*. Given a mixed graph $D = (V, A)$, we call a q -coloring f of D *strictly-compatible* if for every unoriented edge the endpoints have different colors, and for every oriented edge the color of the origin is less than the color of the target. We denote $\chi_D(q)$ the number of strictly-compatible q -colorings of the mixed graph D . This invariant was already studied in [25, 26, 7]. Note that for an unoriented graph G , the strictly-compatible q -colorings are the *proper* q -colorings, so that $\chi_G(q)$ is the chromatic polynomial, which is known to be related to the Tutte polynomial by $\chi_G(q) = (-1)^{|V|-c(D)} q^{c(D)} T_G(1-q, 0)$.

Proposition 3.11. Let D be a mixed graph and let q be a positive integer. Then

$$\chi_D(q) = (-1)^{|V|-c(D)} q^{c(D)} T_D^{(1)}(1-q, 0) = (-1)^{|V|-c(D)} q^{c(D)} T_D^{(2)}(1-q, 0).$$

In particular, the polynomials $T_D^{(1)}(x, 0)$ and $T_D^{(2)}(x, 0)$ are equal, and $\chi_D(q)$ is a polynomial.

Proof. Let $E(D)$ be the arc-partition of D and let $e = |E(D)|$. Observe that for any q -coloring f , one has $|f_A^>| \leq e$, with equality if and only if f is strictly-compatible. This is because in any pair of opposite arcs $\{a, -a\}$ in $E(D)$, at most one arc can be in $f_A^>$. Thus,

$$\begin{aligned} \chi_D(q) &= [y^e] B_D(q, y, 1) = [y^0] y^e B_D(q, 1/y, 1) = [y^0] (y-1)^{|V|-c(D)} q^{c(D)} T_D^{(1)} \left(\frac{q}{y-1} + 1, y \right), \\ &= (-1)^{|V|-c(D)} q^{c(D)} T_D^{(1)}(1-q, 0). \end{aligned}$$

Next, we observe that any strictly-compatible coloring of D is a strictly-compatible coloring of a unique complete orientation \vec{D} of D . Thus,

$$\chi_D(q) = \sum_{\vec{D} \in \text{Orient}(D)} [y^e] B_{\vec{D}}(q, y, 1) = [y^0] \frac{y^e}{2^e} \sum_{\vec{D} \in \text{Orient}(D)} B_{\vec{D}} \left(q, \frac{2-y}{y}, 1 \right) = (-1)^{|V|-c(D)} q^{c(D)} T_D^{(2)}(1-q, 0).$$

□

Remark 3.12. We will see that the invariants $T_D^{(1)}$ and $T_D^{(2)}$ share many features with the Tutte polynomial of graphs. Alas, unlike the Tutte polynomial of graphs, the coefficients of $T_D^{(1)}$ and $T_D^{(2)}$ are not always integers, nor positive as seen in the next example.

Example 3.13. For the digraph D with a single arc represented in Figure 1 (thought as a mixed graph with no unoriented edges), one finds $T_D^{(1)}(x, y) = (xy + x - y)/2$ and $T_D^{(2)}(x, y) = x/2$. For

the digraph D'' represented in Figure 1 (thought as a mixed graph with one unoriented edges), one finds

$$T_{D''}^{(1)}(x, y) = \frac{x^2 y^3 + 2x^2 y^2 - 2xy^3 + 2x^2 y - xy^2 + y^3 + x^2 - xy - y^2 + x + y}{6}, \quad (15)$$

$$T_{D''}^{(2)}(x, y) = \frac{-x^2 y^2 + 2x^2 y - xy^2 + 2x^2 + 2xy + 2y^2 + 2x + 2y}{12}. \quad (16)$$

4. RECURRENCE AND THE TUTTE ACTIVITIES OF MIXED GRAPHS

In this section, we show that the B -polynomial admits a Tutte-like recurrence with respect to *unoriented edges* (that is, pairs of opposite arcs). This allows one to define some notions of partial Tutte-activities. However, no proper recurrence relation holds with respect to single arcs, so the recurrence perspective plays a much lesser role for the B -polynomial than for the Tutte polynomial.

We first extend to digraphs, the known recurrence relation for the Potts function of graphs.

Lemma 4.1. *Let $D = (V, A)$ be a digraph. If two opposite arcs $a = (u, v)$ and $-a = (v, u)$ both belong to A , then*

$$B_D(q, y, z) = (yz)B_{D \setminus e}(q, y, z) + (1 - yz)B_{D/e}(q, y, z), \quad (17)$$

where $e = \{a, -a\}$. Note that in the special case where $u = v$, this gives $B_D(q, y, z) = B_{D/e}(q, y, z)$.

Proof. Let q be a positive integer. For any q -coloring f of D ,

$$y^{|f_A^>|} z^{|f_A^<|} = (yz + (1 - yz) \mathbf{1}_{f(u)=f(v)}) \left(y^{|f_{A \setminus e}^>|} z^{|f_{A \setminus e}^<|} \right).$$

Thus,

$$\begin{aligned} B_D(q, y, z) &= yz \sum_{f: V \rightarrow [q]} y^{|f_{A \setminus e}^>|} z^{|f_{A \setminus e}^<|} + (1 - yz) \sum_{f: V \rightarrow [q]} \mathbf{1}_{f(u)=f(v)} y^{|f_{A \setminus e}^>|} z^{|f_{A \setminus e}^<|} \\ &= yz B_{D \setminus e}(q, y, z) + (1 - yz) B_{D/e}(q, y, z). \end{aligned}$$

□

Next, we establish a relation between B_D , $B_{D \setminus a}$, $B_{D/a}$, and $B_{D^{-a}}$.

Lemma 4.2. *Let $D = (V, A)$ be a digraph. For any arc $a = (u, v)$ in A ,*

$$B_D(q, y, z) + B_{D^{-a}}(q, y, z) = (y + z)B_{D \setminus a}(q, y, z) + (2 - y + z)B_{D/a}(q, y, z). \quad (18)$$

Note that in the special case where $u = v$, this gives $2B_D(q, y, z) = 2B_{D/a}(q, y, z)$.

Proof. Let q be a positive integer. For any q -coloring f of D ,

$$y^{|f_A^>|} z^{|f_A^<|} + y^{|f_{A \cup \{-a\} \setminus a}^>|} z^{|f_{A \cup \{-a\} \setminus a}^<|} = (y + z + (2 - y - z) \mathbf{1}_{f(u)=f(v)}) \left(y^{|f_{A \setminus a}^>|} z^{|f_{A \setminus a}^<|} \right).$$

Thus,

$$\begin{aligned} B_D(q, y, z) + B_{D^{-a}}(q, y, z) &= (y + z) \sum_{f: V \rightarrow [q]} y^{|f_{A \setminus a}^>|} z^{|f_{A \setminus a}^<|} + (2 - y - z) \sum_{f: V \rightarrow [q]} \mathbf{1}_{f(u)=f(v)} y^{|f_{A \setminus a}^>|} z^{|f_{A \setminus a}^<|} \\ &= (y + z)B_{D \setminus a}(q, y, z) + (2 - y - z)B_{D/a}(q, y, z). \end{aligned}$$

□

Note that, unlike Equation (17), Equation (18) does not express the B -polynomial of D in terms of the B -polynomial of digraphs with fewer edges. In fact, the B -polynomial is not the only polynomial to satisfy relation (18) (see for instance Section 9), and there is no proper notion of “universality” with respect to this type of recurrence relation. However, as we now explain, it is still possible to use (17) and (18) to define partial notions of *Tutte activities* by focusing on unoriented edges.

Let us first rewrite Lemmas 4.1 and 4.2 for the invariant $T_D^{(1)}$ and $T_D^{(2)}$ of mixed graphs. We call *bridge* of a mixed graph, an edge whose deletion increases the number of connected components.

Corollary 4.3. *Let D be a mixed graph, and let $e = \{a, -a\} \in E(D)$ be an unoriented edge. For $i \in \{1, 2\}$,*

- *if e is neither a bridge nor a loop, then*

$$T_D^{(i)}(x, y) = T_{D \setminus e}^{(i)}(x, y) + T_{D/e}^{(i)}(x, y).$$

- *if e is a bridge, then*

$$T_D^{(i)}(x, y) = (x - 1)T_{D \setminus e}^{(i)}(x, y) + T_{D/e}^{(i)}(x, y).$$

- *if e is a loop, then*

$$T_D^{(i)}(x, y) = yT_{D \setminus e}^{(i)}(x, y).$$

The proof of Corollary 4.3 for $i = 1$ (resp. $i = 2$) is immediate from Lemmas 4.1 (resp. Lemma 4.2). We now give expressions for $T_D^{(1)}$ and $T_D^{(2)}$ obtained by iterating Corollary 4.3 on the set of unoriented edges.

Proposition 4.4. *Let $D = (V, A)$ be a mixed graph, and let H be the set of unoriented edges. Then for $i \in \{1, 2\}$,*

$$T_D^{(i)}(x, y) = \sum_{R \uplus S = H} (x - 1)^{c(D \setminus R) - c(D)} (y - 1)^{|S| + c(V, S) - |V|} T_{D \setminus R/S}^{(i)}(x, y),$$

where the sum is over all partitions of H into two disjoint sets R, S (there are $2^{|H|}$ summands), and $c(V, S)$ is the number of connected components of the graph (V, S) .

Proof. We apply the recurrence relation of Corollary 4.3 successively on every edge of H in an arbitrary order, with the following twist: when applying the recurrence on a loop e we use the identity

$$T_D^{(i)}(x, y) = T_{D \setminus e}^{(i)}(x, y) + (y - 1)T_{D/e}^{(i)}(x, y). \quad (19)$$

In Figure 4, we represented the process of applying this recurrence successively on every edge of H by a *computation tree*. The root of the computation tree is D , and the leaves of the computation tree correspond to all the digraphs $T_{D \setminus R/S}^{(i)}(x, y)$ with $R \uplus S = H$. The recurrence gives

$$T_D^{(i)}(x, y) = \sum_{R \uplus S = H} (x - 1)^{\alpha(R, S)} (y - 1)^{\beta(R, S)} T_{D \setminus R/S}^{(i)}(x, y),$$

where $\alpha(R, S)$ is the number of bridges deleted during the deletion-contraction process leading from D to $D \setminus R/S$, and $\beta(R, S)$ is the number of loops contracted during this process. Moreover $\alpha(R, S) = c(D \setminus R) - c(D)$ since this number starts at 0 and increases by one exactly when deleting a bridge during the deletion-contraction process (where R represent the set of edges which have been deleted). Similarly $\beta(R, S) = |S| + c(V, S) - |V|$ because this quantity starts at 0 and increases by one exactly when contracting a loop during the deletion-contraction process (where S represent the set of edges which have been contracted). \square

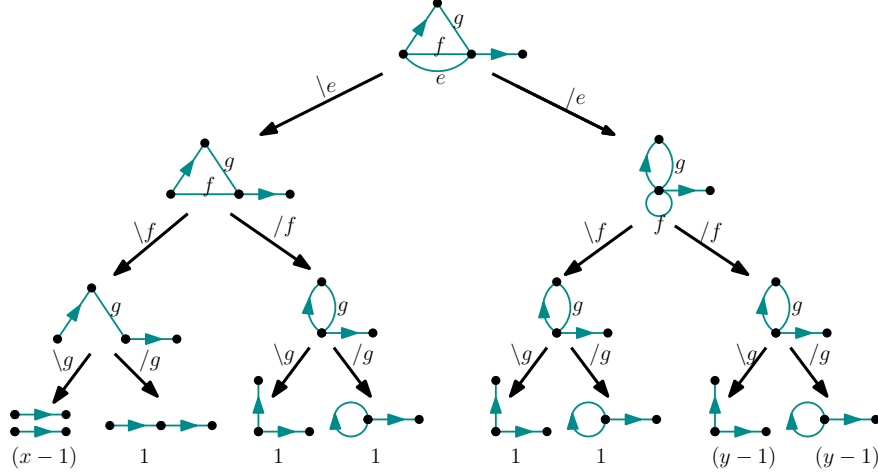


FIGURE 4. The computation tree illustrating the computation of $T_D^{(i)}$ using the recurrence relation (19) for loops. In this figure, the complete edges e, f, g are represented by one segment rather than by two opposite arcs. The factor $(x-1)^{c(D_{\setminus R})-c(D)}(y-1)^{|S|+c(V,S)-|V|}$ is indicated under each of the digraphs $D_{\setminus R/S}$ represented at the bottom of the computation tree.

Proposition 4.5. *Let $D = (V, A)$ be a mixed graph, let H be the set of unoriented edges, and let \prec be a total order on the set H . Then for i in $\{1, 2\}$,*

$$T_D^{(i)}(x, y) = \sum_{F \subseteq H \text{ forest}} (x-1)^{c(S)-c(D)} y^{\text{ext}_{\prec}(F)} T_{D_{\setminus \overline{F}/F}}^{(i)}(x, y),$$

where the sum is over all forests (that is, subset $F \subseteq H$ such that the graph (V, F) has no cycle), and for a forest F we denote $\overline{F} = H \setminus F$, and we denote $\text{ext}_{\prec}(F)$ the number of edges $e \in \overline{F}$ such that there is a path P in F between the endpoints of e and e is smaller than any edge in P for the order \prec .

Proof. We apply the recurrence relation of Corollary 4.3 successively on every edge of H in the decreasing order given by \prec (that is, we start with the largest element). In Figure 5, we represented the process of applying this recurrence successively on every edge of H by a *computation tree*. Since we use the relation $T_D^{(i)}(x, y) = y T_{D_{\setminus e}}^{(i)}(x, y)$ for loops, we are never contracting loops during this process. Hence the set $S \subseteq H$ of edges contracted during this process must be a forest (and this is the only condition on S). Therefore,

$$T_D^{(i)}(x, y) = \sum_{R \sqcup S = H, S \text{ forest}} (x-1)^{\alpha(R, S)} y^{\gamma(R, S)} T_{D_{\setminus R/S}}^{(i)}(x, y),$$

where $\alpha(R, S)$ is the number of bridges deleted during the deletion-contraction process leading from D to $D_{\setminus R/S}$, and $\gamma(R, S)$ is the number of loops deleted during this process. Moreover, $\alpha(R, S) = c(D_{\setminus R}) - c(D)$ since this number starts at 0 and increases by one exactly when deleting a bridge during the deletion-contraction. Similarly, $\gamma(S) = \text{ext}_{\prec}(S)$ because an edge e is deleted as a loop if and only if there is a path of edges in H joining the endpoints of e which is contracted in the deletion-contraction process before the deletion of e . \square

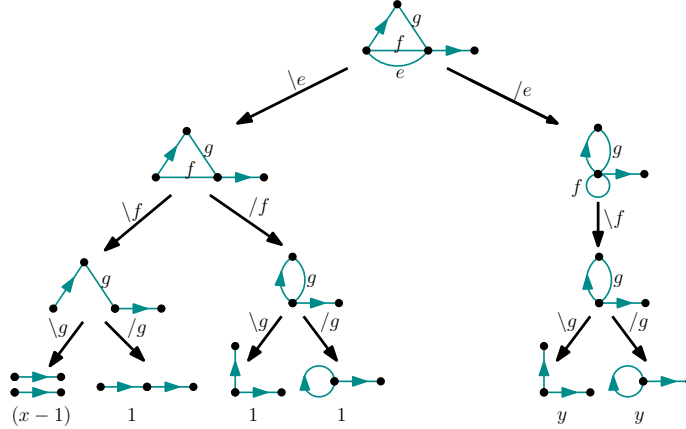


FIGURE 5. The computation tree illustrating the computation of $T^{(i)}$ using the recurrence relation of Corollary 4.3 for loops. The order on $H = \{e, f, g\}$ is $e \succ f \succ g$. The digraph at the bottom of the computation tree are of the form $D_{D \setminus \overline{F}/F}$ with $F \subseteq H$ such that the graph (V, F) has no cycle. The factor $(x-1)^{c(D \setminus \overline{F}) - c(D)} y^{\text{ext}_{\prec}(F)}$ is indicated under each of the digraphs $D_{D \setminus \overline{F}/F}$ at the bottom of the computation tree. For instance, the factor y for the two rightmost digraphs correspond to the edge f (with $P = \{e\}$).

Remark 4.6. In the case where the mixed graph D has no oriented edges (i.e. $D = \overleftrightarrow{G}$ for a graph G), Proposition 4.4 becomes the *subgraph expansion* (6) of the Tutte polynomial:

$$T_D^{(i)}(x, y) = \sum_{S \subseteq E} (x-1)^{c(S) - c(D)} (y-1)^{|S| + c(S) - |V|},$$

while Proposition 4.5 gives the *forest expansion* of the Tutte polynomial:

$$T_D^{(i)}(x, y) = \sum_{F \subseteq E, \text{ forest}} (x-1)^{c(S) - c(D)} y^{\text{ext}_{\prec}(F)}.$$

The forest expansion is given with external activities corresponding to a total order of the arcs of D . As for the Tutte polynomial of graphs, it would be possible to give a more general definition of “external activity with respect to a computation tree” in the sense of Gordon and McMahon [15]. It would also be possible to interpolate between Proposition 4.4 and Proposition 4.5 by mimicking the *generalized activity* construction of Gordon and Traldi [16].

5. ORIENTED CHROMATIC POLYNOMIALS, AND THEIR GENERATING FUNCTIONS

In this section, we define *chromatic polynomials for digraphs*. We then express in terms of the B -polynomial several generating functions of chromatic-polynomials of modified digraphs. These expressions will be used in the subsequent sections in order to extract valuable information from the B -polynomial.

Recall that the *chromatic polynomial* of a graph $G = (V, E)$ is the polynomial $\chi_G(q)$, which for all positive integers q counts the number of proper q -colorings of G :

$$\chi_G(q) = [y^{|E|}] P_G(q, y) = |\{f : V \rightarrow [q] \mid \forall \{u, v\} \in E, f(u) \neq f(v)\}|.$$

We now define some analogous polynomials $\chi_D^>$ and χ_D^{\geq} for digraphs.

Definition 5.1. Let $D = (V, A)$ be a digraph. The *strict-chromatic polynomial* of D , denoted $\chi_D^>(q)$, is the unique polynomial whose evaluation at any positive integer q gives the number of q -colorings of D with only strict ascents:

$$\chi_D^>(q) = [y^{|A|}]B_D(q, y, 1) = |\{f : V \rightarrow [q] \mid \forall (u, v) \in A, f(u) < f(v)\}|.$$

The *weak-chromatic polynomial* of D , denoted $\chi_D^{\geq}(q)$, is the unique polynomial whose evaluation at any positive integer q gives the number of q -colorings of D with only weak ascents:

$$\chi_D^{\geq}(q) = B_D(q, 0, 1) = |\{f : V \rightarrow [q] \mid \forall (u, v) \in A, f(u) \leq f(v)\}|.$$

Remark 5.2. The weak and strict-chromatic polynomials are very closely related to the *order polynomials* of posets, as defined by Stanley in [28, chap. IV] (see also [27]). Indeed, if D is acyclic, then we can consider the partial ordering of vertices \preceq_D , where $u \preceq_D v$ means that there exists a directed path from u to v . In this case, our strict and weak-chromatic polynomials $\chi_D^>(q)$ $\chi_D^{\geq}(q)$ coincide with the weak and strict-order polynomials associated to the poset (V, \preceq_D) .

Remark 5.3. For any graph G ,

$$\chi_G(q) = \sum_{\vec{G} \in \text{Orient}(G)} \chi_{\vec{G}}^>(q), \quad (20)$$

where the sum is over all digraphs \vec{G} obtained by orienting G . Indeed, for any proper q -coloring f of G there is a unique orientation of G such that f has only strict ascents. Equation (20) is one of the keystones in *inside-out* theory of the chromatic polynomial developed by Beck and Zaslavsky in [8].

Remark 5.4. Because the strict and weak-chromatic polynomials are specializations of the B polynomials, they satisfy the same type of recurrence relations. For example, extracting the coefficient of $y^{|A|}$ in (18) gives a relation between $\chi_D^>$, $\chi_{D-a}^>$, $\chi_{D \setminus a}^>$ and $\chi_{D/a}^>$:

$$\chi_D^>(q) + \chi_{D-a}^>(q) = \chi_{D \setminus a}^>(q) - \chi_{D/a}^>(q).$$

We now state the main result of this section.

Theorem 5.5. *Let $D = (V, A)$ be a digraph. Then,*

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D \setminus R}^>(q) = B_D(q, 1+y, 1+z), \quad (21)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D \setminus R}^{\geq}(q) = (1+y+z)^{|A|} B_D\left(q, \frac{1+y}{1+y+z}, \frac{1+z}{1+y+z}\right), \quad (22)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D/R}^>(q) = B_D(q, y, z), \quad (23)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D/R}^{\geq}(q) = (1+y+z)^{|A|} B_D\left(q, \frac{y}{1+y+z}, \frac{z}{1+y+z}\right), \quad (24)$$

where the sum is over all possible ways of partitioning the arc set A in three subsets.

Proof. For any positive integer q , we have

$$B_D(q, y, z) = \sum_{f: V \rightarrow [q]} \prod_{(u, v) \in A} \theta(f(u) - f(v)). \quad (25)$$

where

$$\theta(n) = \begin{cases} y & \text{if } n < 0, \\ z & \text{if } n > 0, \\ 1 & \text{if } n = 0. \end{cases}$$

We now consider several ways of expressing θ as a sum of three terms:

$$\theta(n) = x_1 + x_2 \mathbb{1}_{n < 0} + x_3 \mathbb{1}_{n > 0}, \text{ for } (x_1, x_2, x_3) = (1, y - 1, z - 1), \quad (26)$$

$$\theta(n) = x_1 + x_2 \mathbb{1}_{n \leq 0} + x_3 \mathbb{1}_{n \geq 0}, \text{ for } (x_1, x_2, x_3) = (y + z - 1, 1 - z, 1 - y), \quad (27)$$

$$\theta(n) = x_1 \mathbb{1}_{n=0} + x_2 \mathbb{1}_{n < 0} + x_3 \mathbb{1}_{n > 0}, \text{ for } (x_1, x_2, x_3) = (1, y, z), \quad (28)$$

$$\theta(n) = x_1 \mathbb{1}_{n=0} + x_2 \mathbb{1}_{n \leq 0} + x_3 \mathbb{1}_{n \geq 0}, \text{ for } (x_1, x_2, x_3) = (1 - y - z, y, z). \quad (29)$$

Each of the expressions (26-29) gives an equation for $B_D(q, y, z)$ which, after a suitable change of variables, give the equations (21-24) respectively. For instance, using (26) gives

$$\begin{aligned} B(q, y, z) &= \sum_{f: V \rightarrow [q]} \prod_{(u, v) \in A} (1 + (y - 1) \mathbb{1}_{f(u) < f(v)} + (z - 1) \mathbb{1}_{f(u) > f(v)}), \\ &= \sum_{f: V \rightarrow [q]} \sum_{R \uplus S \uplus T = A} (y - 1)^{|S|} (z - 1)^{|T|} \mathbb{1}_{S \subseteq f_A^>, \text{ and } T \subseteq f_A^<}, \\ &= \sum_{R \uplus S \uplus T = A} (y - 1)^{|S|} (z - 1)^{|T|} \sum_{f: V \rightarrow [q]} \mathbb{1}_{S \subseteq f_A^>, \text{ and } T \subseteq f_A^<}, \\ &= \sum_{R \uplus S \uplus T = A} (y - 1)^{|S|} (z - 1)^{|T|} \chi_{D_{\setminus R}^>}(q). \end{aligned}$$

This gives (21) by a change of variables. Similarly, using (28) gives

$$\begin{aligned} B(q, y, z) &= \sum_{f: V \rightarrow [q]} \prod_{(u, v) \in A} (\mathbb{1}_{f(v) = f(u)} + y \mathbb{1}_{f(u) < f(v)} + z \mathbb{1}_{f(u) > f(v)}), \\ &= \sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \sum_{f: V \rightarrow [q]} \mathbb{1}_{R \subseteq f_A^=, S \subseteq f_A^>, \text{ and } T \subseteq f_A^<}, \\ &= \sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{/R}^>}(q), \end{aligned}$$

where in the last line we identify the q -colorings of D such that $R \subseteq f_A^=$ with the q -colorings of $D_{/R}$. This gives (23) by a change of variables. Equations (22) and (24) are obtained similarly. \square

Each of the generating functions in Theorem 5.5 involves two operations among deletion, contraction, and reorientation. As specializations, we can get generating function involving a single operation. For deletion we get,

$$\sum_{S \subseteq A} y^{|A \setminus S|} \chi_{D_{\setminus S}^>}(q) = B_D(q, y + 1, 1), \quad (30)$$

$$\sum_{S \subseteq A} y^{|A \setminus S|} \chi_{D_{\setminus S}^{\geq}}(q) = (y + 1)^{|A|} B_D\left(q, \frac{1}{y + 1}, 1\right). \quad (31)$$

For contraction we get,

$$\sum_{S \subseteq A} y^{|A \setminus S|} \chi_{D_{/S}^>}(q) = B_D(q, y, 0), \quad (32)$$

$$\sum_{S \subseteq A} y^{|A \setminus S|} \chi_{D_{/S}^{\geq}}(q) = (y + 1)^{|A|} B_D\left(q, \frac{y}{y + 1}, 0\right).$$

For reorientation we get,

$$\begin{aligned} \sum_{S \subseteq A} \alpha^{|S|} \chi_{D-S}^{\geq}(q) &= [y^{|A|}] B_D(q, y, \alpha y), \\ \sum_{S \subseteq A} \alpha^{|S|} \chi_{D-S}^{\geq}(q) &= (1 + \alpha)^{|A|} B_D\left(q, \frac{1}{1 + \alpha}, \frac{\alpha}{1 + \alpha}\right). \end{aligned}$$

Next, we state an easy lemma about the strict-chromatic polynomial, and explore some immediate consequences.

Lemma 5.6. *Let D be a digraph and let $\ell(D)$ be the maximum number of arcs of a directed path of D (so that $\ell = \infty$ if D contains a directed cycle). Then $\chi_D^{\geq}(q) = 0$ for any positive integer $q \leq \ell$, and $\chi_D^{\geq}(q) > 0$ for any positive integer $q > \ell$. In particular, $\chi_D^{\geq} \neq 0$ if and only if D is acyclic.*

Proof. Let q be a positive integer. If f is a q -coloring with only strict ascents, then the colors must be strictly increasing along any directed path. Hence, $\chi_D^{\geq}(q) = 0$ for all $q \leq \ell$. Conversely, $\chi_D^{\geq}(q) > 0$ for all $q > \ell$, because the q -coloring f which associates to each vertex v the length of the longest directed path of D ending at v , has only strict ascents. \square

We now show that several quantities about D can be obtained from $B_D(q, y, z)$. For a digraph D , we denote $\text{acyc}(D)$ the (acyclic) digraph obtained from D by contracting all the cyclic arcs. Note that the vertices of $\text{acyc}(D)$ correspond to the *strongly connected components* of D .

Corollary 5.7. *Let D be a digraph.*

- (1) *The number α of acyclic arcs of D is $\deg_y(B_D(q, y, 0))$.*
- (2) *The number of strongly-connected components of D is $\deg_q([y^\alpha] B_D(q, y, 0))$.*
- (3) *The maximal length of directed paths in $\text{acyc}(D)$ is $\min\{q \in \mathbb{P} \mid [y^\alpha] B_D(q, y, 0) \neq 0\}$.*

Proof. Let α be the number of acyclic arcs of D , and let $\beta = |A| - \alpha$ be the number of cyclic arcs. Observe that contracting a cyclic arc of a digraph decreases the number of cyclic arcs by one, whereas contracting an acyclic arc does not decrease the number of cyclic arcs. Thus, the minimal number of arcs of D to be contracted in order to obtain an acyclic digraph is β . Thus, (32) and Lemma 5.6 show that $\alpha = \deg_y(B_D(q, y, 0))$, and

$$\chi_{\text{acyc}(D)}^{\geq}(q) = [y^\alpha] B_D(q, y, 0).$$

Moreover, by Lemma 5.6, the maximal length of directed paths in $\text{acyc}(D)$ is $\min\{q \in \mathbb{P} \mid \chi_{\text{acyc}(D)}^{\geq}(q) \neq 0\}$. Moreover, we claim that the number of strongly connected components $v(\text{acyc}(D))$ is the degree of $\chi_{\text{acyc}(D)}^{\geq}(q)$. Indeed, the same reasoning as in the proof of Proposition 3.1 shows that for any digraph $D' = (V', A')$,

$$\chi_{D'}^{\geq}(q) = \sum_{p=1}^{|V'|} c_p(D') \binom{q}{p},$$

where $c_p(D')$ is the number of *surjective* p -colorings of D' with only strict ascents. Moreover, when D' is acyclic, $c_q(D') \neq 0$, so that the degree of $\chi_{D'}^{\geq}$ is $|V'|$. Since $\text{acyc}(D)$ is acyclic, the degree of $\chi_{\text{acyc}(D)}^{\geq}(q)$ is indeed $v(\text{acyc}(D))$. \square

6. THE B -POLYNOMIAL AT NEGATIVE q , VIA EHRHART THEORY

In this section, we use Ehrhart theory in order to study the B -polynomial. We first interpret the polynomials $\chi_D^{\geq}(q)$ and $\chi^{\geq}(q)$ as counting lattice points in a polytope.

Definition 6.1. Let $D = (V, A)$ be a digraph. The *ascent polytope* of D is the polytope $\Delta_D \subset \mathbb{R}^V$ made of the points $(x_v)_{v \in V}$ such that

$$\forall v \in V, 0 \leq x_v \leq 1, \text{ and } \forall (u, v) \in A, x_u \leq x_v.$$

The q -dilation of a region Δ of \mathbb{R}^n , is

$$q\Delta := \{(qx_1, \dots, qx_n) \mid (x_1, \dots, x_n) \in \Delta\}.$$

The *interior* of Δ is denoted Δ° . The following lemma relates $\chi_D^{\geq}(q)$ and $\chi^{\geq}(q)$ to the q -dilation of the ascent polytope.

Lemma 6.2. Let $D = (V, A)$ be a digraph with n vertices, and let $\Delta_D \subset \mathbb{R}^V$ be its ascent polytope. Then, for all positive integers q ,

$$\chi_D^{\geq}(q) = |(q+1)\Delta_D^\circ \cap \mathbb{Z}^V|, \text{ and } \chi_D^{\geq}(q) = |(q-1)\Delta_D \cap \mathbb{Z}^V|. \quad (33)$$

Proof. The points $(x_v)_{v \in V}$ in $(q+1)\Delta_D^\circ \cap \mathbb{Z}^V$ are characterized by

$$\forall v \in V, x_v \in [q], \text{ and } \forall (u, v) \in A, x_u < x_v.$$

Hence they identify with the q -colorings f of D with only strict ascents upon setting $f(v) = x_v$ for all $v \in V$. Similarly, the points $(x_v)_{v \in V}$ in $(q-1)\Delta_D \cap \mathbb{Z}^V$ identify with the q -colorings f of D with only weak ascents upon setting $f(v) = x_v + 1$. \square

Let us now recall the main results of Ehrhart theory.

Theorem 6.3 (Ehrhart's Theorem and Ehrhart-Macdonald reciprocity). Let $\Pi \subset \mathbb{R}^n$ be a polytope with integer vertices. Then, there exists a polynomial $E_\Pi(q)$, called Ehrhart polynomial of Π , such that for any non-negative integer q ,

$$E_\Pi(q) = |q\Pi \cap \mathbb{Z}^n|.$$

Moreover, if the interior Π° is non-empty, then for any positive integer q ,

$$E_\Pi(-q) = (-1)^n |q\Pi^\circ \cap \mathbb{Z}^n|. \quad (34)$$

Example 6.4. Consider the 2-dimensional polytope $\Pi = \{(x, y) \in \mathbb{R}^2 \mid x, y \in [0, 1]\} \subset \mathbb{R}^2$. We have $E_\Pi(q) = (q+1)^2$, so that $E_\Pi(-q) = (q-1)^2 = |q\Pi^\circ \cap \mathbb{Z}^2|$.

We now prove a key lemma, which is a simple extension of a result of Stanley about order polynomials [27].

Lemma 6.5. Let $D = (V, A)$ be a digraph and let q be a positive integer. Then,

$$\chi_D^{\geq}(-q) = (-1)^{v(\text{acyc}(D))} \chi_{\text{acyc}(D)}^{\geq}(q), \quad (35)$$

where $\text{acyc}(D)$ is the (acyclic) digraph obtained from D by contracting all the cyclic arcs.

Proof. First note that for any cyclic arc a of D , $\chi_D^{\geq} = \chi_{D/a}^{\geq}$. This is because for any positive integer q and any q -coloring f with only weak ascents, all the vertices along a directed cycle of D must have the same color. Therefore, $\chi_D^{\geq} = \chi_{\text{acyc}(D)}^{\geq}$. Since $\text{acyc}(D)$ is acyclic, it only remains to prove that for an acyclic digraph $D = (V, A)$,

$$\chi_D^{\geq}(q) = (-1)^{|V|} \chi_D^{\geq}(-q). \quad (36)$$

Let D be acyclic and let $E_D(q)$ be the Ehrhart polynomial of the ascent polytope Δ_D . By Lemma 6.2, we have $\chi_D^{\geq}(q) = |(q-1)\Delta_D \cap \mathbb{Z}^V| = E_D(q-1)$. Since D acyclic, we know from Lemma 5.6 that $\chi_D^{\geq} \neq 0$. Hence, by Lemma (6.2), the ascent polytope Δ_D has non-empty interior. Thus, by Ehrhart-Macdonald reciprocity (34),

$$\chi_D^{\geq}(q) = |(q+1)\Delta_D^\circ \cap \mathbb{Z}^V| = (-1)^{|V|} E_D(-q-1) = (-1)^{|V|} \chi_D^{\geq}(-q).$$

\square

We will now explore the many ramifications of Lemma 6.5 for the B -polynomial. First, we obtain some new generating functions of weak and strict-chromatic polynomials in terms of the B -polynomial.

Theorem 6.6. *Let $D = (V, A)$ be a digraph. Then,*

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{\setminus R}^{-T} \text{ acyclic}}} y^{|S|} z^{|T|} \chi_{D_{\setminus R}^{-T}}^{\geq}(q) = (-1)^{|V|} B_D(-q, 1+y, 1+z), \quad (37)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} (-1)^{v(\text{acyc}(D_{\setminus R}^{-T}))} \chi_{\text{acyc}(D_{\setminus R}^{-T})}^{\geq}(q) = (1+y+z)^{|A|} B_D\left(-q, \frac{1+y}{1+y+z}, \frac{1+z}{1+y+z}\right), \quad (38)$$

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{/R}^{-T} \text{ acyclic}}} y^{|S|} z^{|T|} (-1)^{v(D_{/R}^{-T})} \chi_{D_{/R}^{-T}}^{\geq}(q) = B_D(-q, y, z), \quad (39)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} (-1)^{v(\text{acyc}(D_{/R}^{-T}))} \chi_{\text{acyc}(D_{/R}^{-T})}^{\geq}(q) = (1+y+z)^{|A|} B_D\left(-q, \frac{y}{1+y+z}, \frac{z}{1+y+z}\right). \quad (40)$$

Proof. Equations (37-40) follow respectively from Equations (26-29) by applying Lemma 6.5. \square

Specializing Theorem 6.6 shows that $B_D(-1, y, z)$ contains generating functions for the acyclic and totally cyclic digraphs obtained from D by deletions, contractions, or reorientations.

Theorem 6.7. *Let $D = (V, A)$ be a digraph. Then,*

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{\setminus R}^{-T} \text{ acyclic}}} y^{|S|} z^{|T|} = (-1)^{|V|} B_D(-1, 1+y, 1+z), \quad (41)$$

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{\setminus R}^{-T} \text{ totally cyclic}}} y^{|S|} z^{|T|} (-1)^{c(D_{\setminus R}^{-T})} = (1+y+z)^{|A|} B_D\left(-1, \frac{1+y}{1+y+z}, \frac{1+z}{1+y+z}\right), \quad (42)$$

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{/R}^{-T} \text{ acyclic}}} y^{|S|} z^{|T|} (-1)^{v(D_{/R}^{-T})} = B_D(-1, y, z), \quad (43)$$

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{/R}^{-T} \text{ totally cyclic}}} y^{|S|} z^{|T|} = (-1)^{c(D)} (1+y+z)^{|A|} B_D\left(-1, \frac{y}{1+y+z}, \frac{z}{1+y+z}\right). \quad (44)$$

Proof. Observe that $\chi_D^{\geq}(1) = 1$ for any digraph D . Hence, plugging $q = 1$ in (37) and (39) gives (41) and (43) respectively. Observe also that $\chi_D^{\geq}(1) = \mathbf{1}_{D \text{ has no arcs}}$. Hence, $\chi_{\text{acyc}(D)}^{\geq}(1) = \mathbf{1}_{D \text{ is totally cyclic}}$. Moreover, if D is totally cyclic, $v(\text{acyc}(D)) = c(D)$. Thus, plugging $q = 1$ in (38) and (40) gives (42) and (44) respectively. \square

Remark 6.8. Observe that Theorem 6.7 could also be derived directly from the Equations (26-29) using the following identity:

$$\chi_D^{\geq}(-1) = (-1)^{|V|} \mathbf{1}_{D \text{ acyclic}}, \text{ and } \chi_D^{\geq}(-1) = (-1)^{c(D)} \mathbf{1}_{D \text{ totally cyclic}}. \quad (45)$$

The identity (45), in turn, follows from Lemma 6.5, along with the facts that $\chi_D^{\geq}(1) = 1$, and $\chi_D^{\geq}(1) = \mathbf{1}_{D \text{ has no arcs}}$.

Among the results of Theorem 6.7, Equations (41) and (44) are especially nice because they do not involve any sign on their left-hand side. Equation (41) gives the bivariate generating function of acyclic digraphs obtained from D by deleting and reorienting some arcs. This can be specialized to give univariate generating functions of acyclic digraphs obtained by either operation. For instance, the generating function of acyclic reorientations of D , counted according to the numbers of reoriented arcs is

$$\sum_{\substack{S \subseteq A \\ D^{-S} \text{ acyclic}}} y^{|S|} = (-1)^{|V|} [z^{|A|}] B_D(-1, yz, z). \quad (46)$$

Indeed, (46) is obtained from (41) by replacing y by yz and selecting the coefficient of $z^{|A|}$, and noticing $[z^{|A|}] B_D(-1, 1 + yz, 1 + z) = [z^{|A|}] B_D(-1, yz, z)$.

Example 6.9. Consider the digraph D represented on the left of Figure 6 (acyclic triangle). We computed earlier (see Example 1.1)

$$B_D(q, y, z) = q + q(q-1)(y^2 + z^2 + yz) + \frac{q(q-1)(q-2)}{6}(y^3 + z^3 + 2yz(y+z)),$$

so that $-[z^3] B_D(-1, yz, z) = 1 + 2y + 2y^2 + y^3$. Looking at Figure 6, we see that this matches the generating function $\sum_{S \subseteq A, D^{-S} \text{ acyclic}} y^{|S|}$.

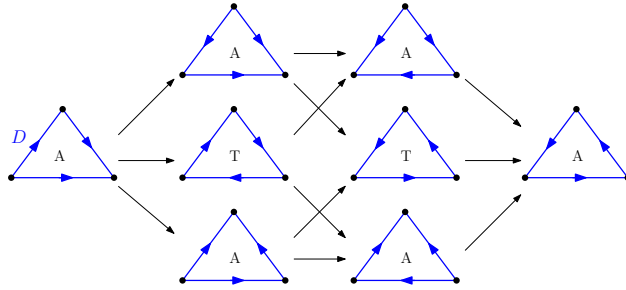


FIGURE 6. A digraph D and its reorientation (organized according to the “graph of arc-flips”). The 6 acyclic reorientations are indicated by a letter A , while the 2 totally cyclic reorientations are marked by a letter T . We see that $\sum_{S \subseteq A, D^{-S} \text{ acyclic}} y^{|S|} =$

$$1 + 2y + 2y^2 + y^3, \text{ and } \sum_{S \subseteq A, D^{-S} \text{ totally cyclic}} y^{|S|} = y + y^2.$$

Equation (41) can also be specialized to give the generating function of acyclic subgraphs of D , counted according to the number of arcs. Indeed, setting $z = 0$ gives

$$\sum_{\substack{S \subseteq A \\ D_{\setminus S} \text{ acyclic}}} y^{|A \setminus S|} = (-1)^{|V|} B_D(-1, 1 + y, 1). \quad (47)$$

Example 6.10. For the digraph D represented on the left of Figure 7 (totally cyclic triangle) we can compute

$$B_D(q, y, z) = q + 3q(q-1)yz + \frac{q(q-1)(q-2)}{2}yz(y+z),$$

so that $(-1)^{|V|} B_D(-1, 1 + y, 1) = 1 + 3y + 3y^2$. Looking at Figure 7, we see that this matches the generating function $\sum_{\substack{S \subseteq A \\ D_{\setminus S} \text{ acyclic}}} y^{|A \setminus S|}$.

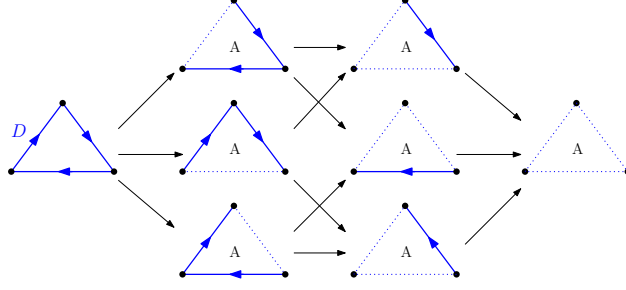


FIGURE 7. A digraph D and its subgraphs. The 7 acyclic subgraphs are indicated by a letter A . We see that $\sum_{S \subseteq A, D_{\setminus S} \text{ acyclic}} y^{|A \setminus S|} = 1 + 3y + 3y^2$.

Equation (44) gives the bivariate generating function of totally cyclic digraphs obtained from D by contracting and reorienting some arcs. In particular, (44) can be specialized to give the generating function of totally cyclic reorientations of D , counted according to the numbers of reoriented arcs:

$$\sum_{\substack{S \subseteq A \\ D^{-S} \text{ totally cyclic}}} y^{|S|} = (-1)^{c(D)} (1+y)^{|A|} B_D \left(-1, \frac{y}{1+y}, \frac{1}{1+y} \right). \quad (48)$$

Indeed, (48) is obtained from (44) by setting $y = yz$, selecting the coefficient of $z^{|A|}$, and noticing

$$[z^{|A|}] (1 + yz + z)^{|A|} B_D \left(-1, \frac{yz}{1 + yz + z}, \frac{z}{1 + yz + z} \right) = (1+y)^{|A|} B_D \left(-1, \frac{y}{1+y}, \frac{1}{1+y} \right).$$

Example 6.11. For the digraph D represented on the left of Figure 6, we can compute

$$(-1)^{c(D)} (1+y)^{|A|} B_D \left(-1, \frac{y}{1+y}, \frac{1}{1+y} \right) = y + y^2.$$

Looking at Figure 6, we see that this matches the generating function $\sum_{\substack{S \subseteq A \\ D^{-S} \text{ totally cyclic}}} y^{|S|}$.

Similarly, setting $z = 0$ in (44) gives the generating function of totally cyclic contractions of D , counted according to the number of arcs:

$$\sum_{\substack{S \subseteq A \\ D_{/S} \text{ totally cyclic}}} y^{|A \setminus S|} = (-1)^{c(D)} (1+y)^{|A|} B_D \left(-1, \frac{y}{1+y}, 0 \right). \quad (49)$$

Example 6.12. For the digraph D represented on the left of Figure 8, we can compute

$$(-1)^{c(D)} (1+y)^{|A|} B_D \left(-1, \frac{y}{1+y}, 0 \right) (-1)^{c(D)} = y^2 + 3y + 1.$$

Looking at Figure 8, we see that this matches the generating function $\sum_{\substack{S \subseteq A \\ D_{/S} \text{ totally cyclic}}} y^{|A \setminus S|}$.

We now explore some duality relations and symmetries implied by Lemma (6.5). Recall that a digraph D is *planar* if it can be drawn in the plane without arc crossings. Let D be a connected planar digraph. A *dual digraph* is a digraph D^* obtained by choosing a planar drawing of D , and then placing a vertex of D^* in each face of D , and drawing an arc a^* of D^* across each arc a of D

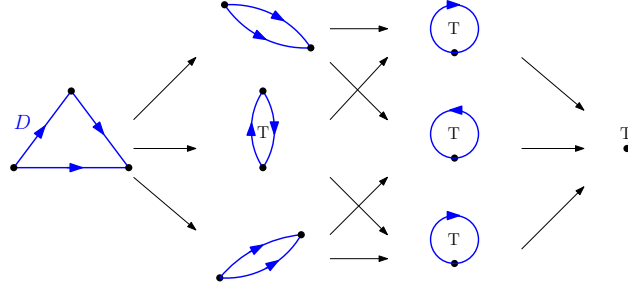


FIGURE 8. A digraph D and its contraction. The 5 totally cyclic contractions are indicated by a letter T . We see that
$$\sum_{S \subseteq A, D_{\setminus S} \text{ totally cyclic}} y^{|A \setminus S|} = y^2 + 3y + 1.$$

with a^* oriented from the left of a to the right of a . See Figure 9. If D is a disconnected planar digraph, then a dual is obtained by applying the preceding procedure to each connected component of D .

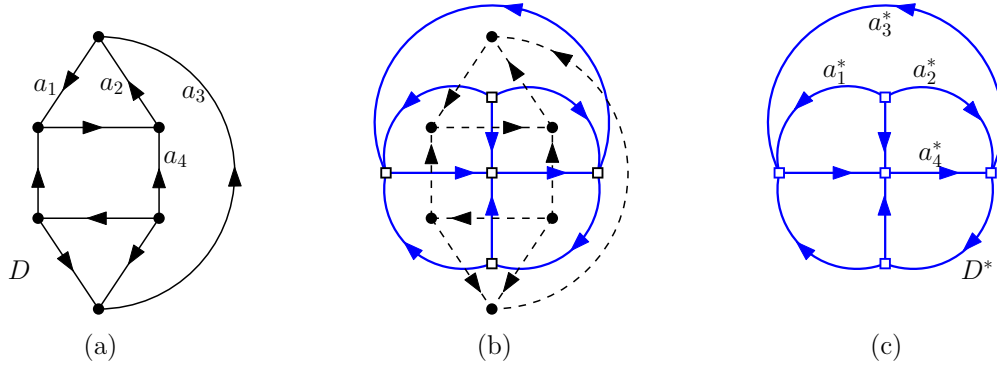


FIGURE 9. (a) A planar digraph. (b) Constructing the dual digraph D^* , by drawing an arc of D^* across each arc of D . (c) The dual digraph D^* .

Theorem 6.13. *Let $D = (V, A)$ be a planar digraph, and let $D^* = (V^*, A^*)$ be a dual digraph. Then the polynomials $B_D(-1, y, z)$ and $B_{D^*}(-1, y, z)$ are related by the following change of variables:*

$$B_{D^*}(-1, y, z) = (-1)^{c(D) - |V|} (1 - y - z)^{|A|} B_D \left(-1, \frac{1 - y}{1 - y - z}, \frac{1 - z}{1 - y - z} \right). \quad (50)$$

Proof. We will explain how to obtain (50) by comparing (41) for D with (44) for D^* . Let $S \subseteq A$ and let $S^* \subseteq A^*$ be the set of arcs dual to S . Note that the dual of D^{-S} is D^{*-S^*} , and the dual of $D_{\setminus S}$ is $D^*_{/S^*}$ (because the dual of deleting an arc is contracting the dual arc). Moreover, a planar digraph D' is acyclic if and only if its dual D'^* is totally cyclic (because the dual of an acyclic arc

is a cyclic arc). Thus using (41) for D and (44) for D^* gives

$$\begin{aligned}
(-1)^{|V|} B_D(-1, 1+y, 1+z) &= \sum_{\substack{R \uplus S \uplus T = A \\ D \setminus_R^{-T} \text{ acyclic}}} y^{|S|} z^{|T|} \\
&= \sum_{\substack{R^* \uplus S^* \uplus T^* = A^* \\ D^* \setminus_{R^*}^{-T^*} \text{ totally cyclic}}} y^{|S|} z^{|T|} \\
&= (-1)^{c(D)} (1+y+z)^{|A|} B_{D^*} \left(-1, \frac{y}{1+y+z}, \frac{z}{1+y+z} \right),
\end{aligned}$$

which is equivalent to (50). \square

Remark 6.14. Recall the classical duality relation for the Tutte polynomial of a planar graph $G = (V, E)$:

$$T_{G^*}(x, y) = T_G(y, x). \quad (51)$$

Using (7) (and the Euler relation $|V| + |V^*| = |A| + 2c(D)$), this translates into the following relation for the Potts polynomial:

$$P_{G^*}(q, y) = \frac{((q-1)y+1)^{|A|}}{q^{|V|-c(G)}} P_G \left(q, \frac{y-1}{(1-q)y-1} \right). \quad (52)$$

Hence, using (11) we obtain another bivariate duality relation for the B -polynomial of a planar digraph $D = (V, A)$:

$$B_{D^*}(q, y, y) = \frac{((q-1)y+1)^{|A|}}{q^{|V|-c(G)}} B_D \left(q, \frac{y-1}{(1-q)y-1}, \frac{y-1}{(1-q)y-1} \right). \quad (53)$$

Although a duality relation holds for the two bivariate specializations $y = z$ and $q = -1$ of the B -polynomial, there cannot be a duality relation for the trivariate B -polynomial. Indeed, the dual of any tree is a digraph with only loops, hence having B -polynomial equal to 1. But since the B -polynomial detects the length of directed paths, there must be infinitely many B -polynomials of trees (even after renormalizing by any prefactor depending on the number of vertices, arcs, and connected components).

Lemma (6.5) also implies some “symmetries” in the coefficients of $B_D(q, y, 1)$ when D is acyclic, and in the coefficients of $B_D(q, y, z)$ when D is a forest.

Theorem 6.15. *If $D = (V, A)$ is an acyclic digraph, then*

$$B_D(-q, y, 1) = (-1)^{|V|} y^{|A|} B_D(q, 1/y, 1). \quad (54)$$

If the underlying graph of D is a forest, then

$$B_D(-q, y, z) = (-1)^{|V|} (y+z-1)^{|A|} B_D \left(q, \frac{y}{y+z-1}, \frac{z}{y+z-1} \right). \quad (55)$$

Proof. If D is acyclic, all its subgraphs are acyclic. Hence, setting $z = 0$ in (37) gives

$$\sum_{R \subseteq A} y^{|A \setminus R|} \chi_{D \setminus_R}^{\geq}(q) = (-1)^{|V|} B_D(-q, y+1, 1).$$

Comparing this to (31) gives $(-1)^{|V|} B_D(-q, y+1, 1) = (y+1)^{|A|} B_D(q, 1/(y+1), 1)$ which is equivalent to (54). If the graph underlying D is a forest, then any digraph obtained from D by deleting or reorienting arcs is acyclic. Hence, (37) gives

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D \setminus_R}^{\geq}(q) = (-1)^{|V|} B_D(-q, 1+y, 1+z).$$

Comparing this to (22) gives $(-1)^{|V|} B_D(-q, 1+y, 1+z) = (1+y+z)^{|A|} B_D\left(q, \frac{1+y}{1+y+z}, \frac{1+z}{1+y+z}\right)$, which is equivalent to (55). \square

We end this section with a question. Note that setting $y = -1$ in (30) gives

$$\sum_{S \subseteq A} (-1)^{|A \setminus S|} \chi_{D \setminus S}^{\geq}(q) = \chi_D^{\geq}(q).$$

Similarly, from (31) we get

$$\sum_{S \subseteq A} (-1)^{|A \setminus S|} \chi_{D \setminus S}^{\geq}(q) = \chi_D^{\geq}(q).$$

These identities can also be obtained by a direct inclusion-exclusion argument. Now, setting $q = -1$ and using (45) gives the following intriguing result.

Proposition 6.16. *For any digraph $D = (V, A)$,*

$$(-1)^{|V| - c(D)} \sum_{\substack{S \subseteq A \\ D \setminus S \text{ acyclic}}} (-1)^{|A \setminus S|} = \mathbf{1}_{D \text{ totally cyclic}}, \quad (56)$$

$$\sum_{\substack{S \subseteq A \\ D \setminus S \text{ totally cyclic}}} (-1)^{|A \setminus S| + c(D \setminus S) - |V|} = \mathbf{1}_{D \text{ acyclic}}. \quad (57)$$

Question 6.17. Is it possible to prove the identities (56) and (57) by a direct combinatorial argument (for instance, by using a “sign reversing involution”)? Note that (57) is obvious when D is acyclic (only 1 totally cyclic subgraph). It is also easy to prove (56) when D is not totally cyclic. Indeed this can be done by considering the involution which “flips” a given acyclic arc of D (adding it if is absent, removing it if it is present). But the other cases seem more challenging.

7. TUTTE POLYNOMIALS AND THEIR EVALUATIONS

In this section, we reinterpret some of the results of the previous sections (those about the specialization $B_D(q, y, 1)$) in terms of the invariants $T^{(1)}$ and $T^{(2)}$ of mixed graphs, and establish some links with the classical theory of the Tutte polynomial.

Let us first express the invariants $T^{(1)}$ and $T^{(2)}$ in terms of the strict and weak chromatic polynomials. Using (30) and (31) we get two expressions for each invariant:

$$T_D^{(1)}(x, y) = \frac{y^{|E|}}{(x-1)^{c(D)}(y-1)^{|V|}} \sum_{S \subseteq A} \left(\frac{1-y}{y}\right)^{|A \setminus S|} \chi_{D \setminus S}^{\geq}((x-1)(y-1)), \quad (58)$$

$$T_D^{(1)}(x, y) = \frac{y^{|E| - |A|}}{(x-1)^{c(D)}(y-1)^{|V|}} \sum_{S \subseteq A} (y-1)^{|A \setminus S|} \chi_{D \setminus S}^{\geq}((x-1)(y-1)), \quad (59)$$

$$T_D^{(2)}(x, y) = \frac{(y/2)^{|E|}}{(x-1)^{c(D)}(y-1)^{|V|}} \sum_{\vec{D} = (V, \vec{E}) \in \text{Orient}(D)} \sum_{S \subseteq \vec{E}} \left(\frac{2(1-y)}{y}\right)^{|\vec{E} \setminus S|} \chi_{\vec{D} \setminus S}^{\geq}((x-1)(y-1)) \quad (60)$$

$$T_D^{(2)}(x, y) = \frac{(1-y/2)^{|E|}}{(x-1)^{c(D)}(y-1)^{|V|}} \sum_{\vec{D} = (V, \vec{E}) \in \text{Orient}(D)} \sum_{S \subseteq \vec{E}} \left(\frac{y-1}{1-y/2}\right)^{|\vec{E} \setminus S|} \chi_{\vec{D} \setminus S}^{\geq}((x-1)(y-1)). \quad (61)$$

We now prove the polynomiality of $T_D^{(1)}$ and $T_D^{(2)}$.

Proposition 7.1. *For any digraph $D = (V, A)$, the invariants $T_D^{(1)}(x, y)$ and $T_D^{(2)}(x, y)$ defined by (13) and (14) are polynomials in x and y .*

Proof. For a digraph D , we denote $\tilde{\chi}_D^>(q) = q^{-c(D)}\chi_D^>(q) = [y^{|A|}]q^{-c(D)}B_D(q, y, 1)$. By Proposition 3.2(i), $q^{c(D)}$ divides $B_D(q, y, z)$, hence $\tilde{\chi}_D^>(q)$ is a polynomial in q . Moreover, (58) gives

$$T_D^{(1)}(x, y) = \sum_{S \subseteq A} (-1)^{|A \setminus S|} (x-1)^{c(D \setminus S) - c(D)} y^{|E| - |A \setminus S|} (y-1)^{|A \setminus S| + c(D \setminus S) - |V|} \tilde{\chi}_{D \setminus S}^>((x-1)(y-1)).$$

The exponent of $(x-1)$ in the above sum is clearly non-negative. The exponent of y is non-negative if $D \setminus S$ is acyclic, which we can assume by Lemma 5.6. The exponent of $(y-1)$ is also non-negative since it corresponds to the *nullity* (a.k.a. cyclomatic number) of the graph underlying $D \setminus S$. Hence $T_D^{(1)}(x, y)$ is a polynomial. The same argument starting from (60) shows that $T_D^{(2)}(x, y)$ is a polynomial. \square

Equations (58) and (60) simplify greatly for $y = 0$ and give

$$T_D^{(1)}(x, 0) = T_D^{(2)}(x, 0) = \frac{(-1)^{|V|}}{(x-1)^{c(D)}} \sum_{\vec{D} \in \text{Orient}(D)} \chi_{\vec{D}}^>(1-x). \quad (62)$$

This could also be obtained directly from Proposition 3.11 upon observing $\chi_D(q) = \sum_{\vec{D} \in \text{Orient}(D)} \chi_{\vec{D}}^>(q)$. Now, using (36) gives

$$(-1)^{|V|} \chi_D(1-x) = (x-1)^{c(D)} T_D^{(1)}(x, 0) = (x-1)^{c(D)} T_D^{(2)}(x, 0) = \sum_{\substack{\vec{D} \in \text{Orient}(D) \\ \vec{D} \text{ acyclic}}} \chi_{\vec{D}}^>(x-1). \quad (63)$$

This gives the following result.

Corollary 7.2 ([7]). *For any mixed graph D ,*

$$(-1)^{|V|} \chi_D(-1) = T_D^{(1)}(2, 0) = T_D^{(2)}(2, 0) = \# \text{ acyclic complete orientations of } D. \quad (64)$$

More generally,

$$(-1)^{|V|} \chi_D(-q) = \sum_{f: V \rightarrow [q]} \gamma_f,$$

where the sum is over all the q -colorings of D , and γ_f is the number of acyclic complete orientations \vec{D} of D such that there is no strict descent of colors along the arcs of \vec{D} .

Proof. Setting $x = 2$ in (63) immediately gives (64) since $\chi_{\vec{D}}^>(x-1) = 1$. Setting $x = q+1$ in (63)

$$(-1)^{|V|} \chi_D(-q) = \sum_{\substack{\vec{D} \in \text{Orient}(D) \\ \vec{D} \text{ acyclic}}} \chi_{\vec{D}}^>(q).$$

Moreover the right-hand side can be interpreted as the number of pairs (\vec{D}, f) , where \vec{D} an acyclic complete orientations of D , and f is a q -coloring such that there is no strict descent of colors along the arcs of \vec{D} . \square

Corollary 7.2 was first proved by Stanley for unoriented graphs [29], and by Beck, Bogart and Pham for mixed graphs [7]².

²Only the invariant χ_D is studied in [7], and not $T_D^{(1)}, T_D^{(2)}$.

Similarly, Equations (61) simplifies for $y = 2$:

$$T_D^{(2)}(x, 2) = \frac{1}{(x-1)^{c(D)}} \sum_{\vec{D} \in \text{Orient}(D)} \chi_{\vec{D}}^{\geq}(x-1). \quad (65)$$

In particular, setting $x = 0$ in (65), and using (45) gives the following result dual to (64).

Corollary 7.3. *For any mixed graph D ,*

$$T_D^{(2)}(0, 2) = \# \text{ totally cyclic complete orientations of } D. \quad (66)$$

This generalizes to mixed graphs the result that Las Vergnas established for unoriented graphs [19].

Example 7.4. Recall that the polynomials $T_{D''}^{(1)}(x, y)$ and $T_{D''}^{(2)}(x, y)$ for the mixed graph D'' of Figure 1 (considered as a mixed graph with one unoriented edge) are given in (15) and (16). We find $T_{D''}^{(1)}(2, 0) = T_{D''}^{(2)}(2, 0) = T_{D''}^{(2)}(0, 2) = 1$ reflecting the fact that there is 1 complete orientation of D which is acyclic and 1 which is totally cyclic.

The similarity between (63) and (65) also gives the following result.

Corollary 7.5. *Let D be a mixed graph such that the unoriented edges form a forest and the digraph obtained by contracting all the unoriented edges is acyclic. Then*

$$T_D^{(2)}(x, 2) = T_D^{(2)}(x, 0).$$

Proof. It is not hard to see that the condition on D implies that all the complete orientations of D are acyclic (in fact, the two properties are equivalent). Hence comparing (63) and (65) immediately gives the result. \square

In the special case where D is acyclic we have additional symmetries. Indeed, (54) translates into the following result.

Corollary 7.6. *If D is an acyclic digraph (seen as a mixed graph with no unoriented edges), then*

$$y^{|A|+c(D)-|V|} T_D^{(1)}(1 + (x-1)y, 1/y) = T_D^{(1)}(x, y), \quad (67)$$

$$T_D^{(2)}(x, 2-y) = T_D^{(2)}(x, y). \quad (68)$$

We will now generalize (64). Translating (47) in terms of $T^{(1)}$ gives the following result.

Theorem 7.7. *For any mixed graph, the polynomial $T_D^{(1)}(x, y)$ contains the generating function of the acyclic subgraphs of D , counted according to the number of arcs:*

$$(y+1)^{|E|+c(D)-|V|} T_D^{(1)}\left(y+2, \frac{y}{y+1}\right) = \sum_{S \subseteq A, D \setminus S \text{ acyclic}} y^{|E|-|A \setminus S|}. \quad (69)$$

Example 7.8. For the mixed graph D'' of Figure 1, one finds $(y+1)T_{D''}^{(1)}\left(y+2, \frac{y}{y+1}\right) = y^3 + 4y^2 + 5y + 1$ reflecting the fact that the number of acyclic subgraphs of D with 0 arc (resp. 1 arc, 2 arcs, 3 arcs) is 1 (resp. 4, 5, 1).

Note that the special case $y = 0$ of Theorem (7.7) is (64). The case of Theorem (7.7) corresponding to unoriented graphs was first proved by Backman in [4], and extended as part of the theory of *fourientations* developed in [5, 6]. Recall that in a *fourientation* of a graph G , the edges of G can be oriented in either direction (1-way edges), in both direction (2-way edges), or in no direction (0-way edges). So *fourientations* of a graph G are naturally identified with the subgraphs of the digraph \overleftrightarrow{G} .

In [4] it is proved that

$$(y+1)^{|E|+c(D)-|V|} T_G(y+2, y/(y+1)) = \sum_{\text{acyclic fourorientations}} y^{\#0\text{-way edges}},$$

which coincides with (69) for the digraph $D = \overleftrightarrow{G}$. The paper [4] also contains the following dual statement:

$$(y+1)^{|V|-c(D)} T_G(y/(y+1), y+2) = \sum_{\substack{\text{totally cyclic fourorientations} \\ \text{without 0-way edge}}} y^{\#2\text{-way edges}}. \quad (70)$$

This result can be recovered from (44) as follows. Setting $z = -1$ in (44) gives

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{/R}^{-T} \text{ totally cyclic}}} y^{|S|} (-1)^{|T|} = (-1)^{c(D)} y^{|A|} B_D(-1, -1/y, 1),$$

which translates into

$$y^{|A|-|E|} (y-1)^{|V|-c(D)} T_D^{(1)}\left(\frac{y-2}{y-1}, y\right) = \sum_{\substack{R \uplus S \uplus T = A \\ D_{/R}^{-T} \text{ totally cyclic}}} y^{|S|} (-1)^{|R|}. \quad (71)$$

Now suppose $D = (V, A) = \overleftrightarrow{G}$ is a graph. To a partition $R \uplus S \uplus T = A$, we associate the fourorientation where an edge $e = \{a, -a\}$ of G is oriented 1-way if one of the arcs $a, -a$ is in S and the other is in T , and oriented 2-way in all the other cases. The 7 different configurations where the edge e would be 2-ways in the associated fourorientation are represented in Table 1, along with their contribution to (71).

Set containing a	R	R	R	S	S	T	T
Set containing $-a$	R	S	T	R	S	R	T
Contribution to (71)	1	$-y$	-1	$-y$	y^2	-1	1

TABLE 1. The configurations such that the edge $e = \{a, -a\}$ is 2-way in the associated fourorientation. The total contribution to the sum in the right-hand side of (71) is a factor of $1 - y - 1 - y + y^2 - 1 + 1 = y^2 - 2y$.

Observe that $D_{/R}^{-T}$ is totally cyclic if and only if the associated fourorientation of G is totally cyclic. Thus, (71) gives

$$y^{|E|} (y-1)^{|V|-c(D)} T_{\overleftrightarrow{G}}^{(1)}\left(\frac{y-2}{y-1}, y\right) = \sum_{\substack{\text{totally cyclic fourorientations} \\ \text{without 0-way edge}}} y^{\#1\text{-way edges}} (y^2 - 2y)^{\#2\text{-way edges}},$$

which is equivalent to (70).

We end this section by mentioning two mysterious identities which could deserve further investigation. We know that for any graph $G = (V, E)$, $T_{\overleftrightarrow{G}}^{(1)}(0, 2) = T_G(0, 2)$ is the number of totally cyclic orientations of G . Hence, specializing (59) to $(x, y) = (0, 2)$, using (45) gives

$$2^{-|E|} \sum_{\substack{S \subseteq \overleftrightarrow{E} \\ \overleftrightarrow{G}_{\setminus S} \text{ totally cyclic}}} (-1)^{c(D_{\setminus S})-c(D)} = \# \text{ totally cyclic orientations of } G, \quad (72)$$

where \overleftrightarrow{E} is the arc-set of \overleftrightarrow{G} . Similarly, specializing (58) to $(x, y) = (0, 2)$, using (45) gives

$$2^{|E|}(-1)^{|V|-c(D)} \sum_{\substack{S \subseteq \overleftrightarrow{E} \\ \overleftrightarrow{G}_{\setminus S} \text{ acyclic}}} (-1/2)^{|\overleftrightarrow{E} \setminus S|} = \# \text{ totally cyclic orientations of } G. \quad (73)$$

Remark 7.9. Equations (72) and (73) can be interpreted in terms of the fourorientations of G . Equation (73) can be written as

$$2^{|E|}(-1)^{|V|-c(D)} \sum_{\text{acyclic fourorientation of } G \text{ without 2-way edges}} (-1/2)^{\# \text{ 1-way edge}} (1/4)^{\# \text{ 0-way edges}} = T_G(0, 2).$$

and this follows easily from the results in [4]. As for (72), Sam Hopkins communicated to us the following alternative proof. By setting $\lambda = 1/2$, $\xi = -2$, $x = -2$, and $y = -1/2$ in Kung's convolution-multiplication formula [18, Identity 3], one gets

$$2^{-|E|} \sum_{S \subseteq E(G)} (-1)^{c(G_{\setminus S})-c(G)} 2^{|V|-c(G_{\setminus S})} T_{G_{\setminus S}}(1/2, 3) = T_G(0, 2), \quad (74)$$

by using the fact that for any graph $H = (U, F)$, $T_H(-1, 1/2) = (-1)^{|U|-c(H)} 2^{|U|-c(H)-|F|}$. Then, from the fourorientation formula of [6], one gets that for any graph $H = (U, F)$, $2^{|U|-c(H)} T_H(1/2, 3)$ is the number α_H of totally cyclic fourorientations of H without 0-way edges. Plugging this result in (74) gives

$$2^{-|E|} \sum_{S \subseteq E} (-1)^{c(G_{\setminus S})-c(G)} \alpha_{G_{\setminus S}} = T_G(0, 2),$$

which is equivalent to (72). Although this proof bypasses our use of Ehrhart theory, it is rather indirect and relies on the deep results of [6] and [18]. Hence, one would hope that a more direct combinatorial explanation of (72) and (73) exists.

Question 7.10. Is it possible to prove the identities (72) and (73) by a direct combinatorial argument?

8. A QUASISYMMETRIC FUNCTION GENERALIZATION OF THE B -POLYNOMIAL

In this section, we study a refinement of the B -polynomial with infinitely many variables $\{x_i\}_{i \in \mathbb{P}}$. This invariant counts colorings by ascents and descents, but also records the number of vertices colored i for all $i \in \mathbb{P}$. As will be clear from the definition, this invariant is quasisymmetric in the variables $\{x_i\}_{i \in \mathbb{P}}$, and we study its expansion in the monomial and fundamental bases of quasisymmetric functions.

Let $\{x_i\}_{i \in \mathbb{P}}$ be indeterminates, and let us denote $\mathbf{x} = (x_1, x_2, x_3, \dots)$. For a digraph $D = (V, A)$, we define

$$B_D(\mathbf{x}; y, z) = \sum_{f: V \rightarrow \mathbb{P}} \left(\prod_{v \in V} x_{f(v)} \right) y^{|f_A^>|} z^{|f_A^<|}.$$

We call this invariant the *quasisymmetric B -polynomial*. It is clear that for any positive integer q ,

$$B_D(1^q; y, z) = \sum_{f: V \rightarrow [q]} y^{|f_A^>|} z^{|f_A^<|} = B_D(q, y, z), \quad (75)$$

where 1^q denotes the infinite sequence $(1, 1, \dots, 1, 0, 0, \dots)$, where the first q coordinates are 1 and the remaining ones are 0. Hence $B_D(\mathbf{x}; y, z)$ clearly determines $B_D(q, y, z)$.

We now recall several specializations of $B_D(\mathbf{x}; y, z)$ appearing in the literature. Recall that for a graph $G = (V, E)$, the *chromatic symmetric function* defined by Stanley in [30] is

$$X_G(\mathbf{x}) = \sum_{\substack{f: V \rightarrow \mathbb{P} \\ f_{\bar{E}} = \emptyset}} \prod_{v \in V} x_{f(v)},$$

and the *Tutte symmetric function* defined by Stanley in [31] is

$$S_G(\mathbf{x}; y) = \sum_{f: V \rightarrow \mathbb{P}} \left(\prod_{v \in V} x_{f(v)} \right) (1 + y)^{|f_{\bar{E}}|}.$$

The Tutte symmetric function is also equivalent to the U -polynomial defined by Noble and Welsh in [23]. Recall also that in [24] Shareshian and Wachs defined the *chromatic quasisymmetric function* of a digraph³ $D = (V, A)$ as

$$\mathcal{X}_D(\mathbf{x}; y) = \sum_{\substack{f: V \rightarrow \mathbb{P} \\ f_{\bar{A}} = \emptyset}} \left(\prod_{v \in V} x_{f(v)} \right) y^{f_{\bar{A}}}. \quad (76)$$

Clearly $S_G(\mathbf{x}; -1) = X_G(\mathbf{x})$ for any graph G , and $\mathcal{X}_D(\mathbf{x}; 1) = X_{\underline{D}}(\mathbf{x})$ for any digraph D . Moreover, the Tutte symmetric function generalizes the Potts polynomial because

$$P_G(q, y) = \sum_{f: V \rightarrow [q]} y^{|f_{\bar{E}}|} = y^{|E|} S_G \left(1^q; \frac{1}{y} - 1 \right).$$

We now explain the relations between these invariants and the quasisymmetric B -polynomial. First, it follows directly from the definitions that

$$\mathcal{X}_D(\mathbf{x}; y) = [z^{|A|}] B_D(\mathbf{x}; yz, z). \quad (77)$$

Moreover, the relations between the B -polynomial and the Potts polynomial given in Propositions (8), (10) and (11) can easily be lifted to relations between $B_D(\mathbf{x}; y, z)$ and $S_G(\mathbf{x}, y)$:

Theorem 8.1. *For any graph $G = (V, E)$,*

$$B_{\vec{G}}(\mathbf{x}; y, z) = (yz)^{|E|} S_G \left(\mathbf{x}; \frac{1}{yz} - 1 \right), \quad (78)$$

$$\frac{1}{2^{|E|}} \sum_{\vec{G} \in \text{Orient}(G)} B_{\vec{G}}(\mathbf{x}; y, z) = \left(\frac{y+z}{2} \right)^{|E|} S_G \left(\mathbf{x}; \frac{2}{y+z} - 1 \right). \quad (79)$$

Moreover, for any digraph D ,

$$B_D(\mathbf{x}; y, y) = y^{|E|} S_{\underline{D}} \left(\mathbf{x}; \frac{1}{y} - 1 \right). \quad (80)$$

Proof. The proofs of Propositions 3.5, 3.6 and (3.7) extend verbatim to prove (78), (79) and (80). \square

Equations (77) and (78) show that the quasisymmetric B -polynomial is a common generalization of the Tutte symmetric function (defined for digraphs) and the chromatic quasisymmetric function (defined for acyclic digraphs).

In order to state other properties of $B_D(\mathbf{x}; y, z)$, we need to recall some basic definitions and results about quasisymmetric functions. Let R be a ring. A *quasisymmetric function in \mathbf{x}* with

³Shareshian and Wachs actually consider labeled graphs instead of digraphs in [24]. Our equivalent definition (76) is given in terms of the (acyclic) digraph D obtained from the labeled graph by orienting every edge from the endpoint of smaller label to the endpoint of larger label.

coefficient in R is a formal power series f in the variables $\{x_i\}_{i \in \mathbb{P}}$ with coefficients in R , such that the degrees of the monomials occurring in f are bounded, and for all positive integers k , $\delta_1, \dots, \delta_k$, $i_1 < i_2 < \dots < i_k$ and $j_1 < j_2 < \dots < j_k$,

$$[x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \dots x_{i_k}^{\delta_k}] f = [x_{j_1}^{\delta_1} x_{j_2}^{\delta_2} \dots x_{j_k}^{\delta_k}] f. \quad (81)$$

A *symmetric function* is a quasisymmetric function such that (81) holds for any tuples (j_1, j_2, \dots, j_k) of distinct integers (not necessarily increasing). The set of quasisymmetric functions (resp. symmetric function) in \mathbf{x} with coefficients in R is denoted $\text{QSym}_R(\mathbf{x})$ (resp. $\text{Sym}_R(\mathbf{x})$), and has the structure of a R -algebra. We denote $\text{QSym}_R^n(\mathbf{x})$ (resp. $\text{Sym}_R^n(\mathbf{x})$) the submodule of $\text{QSym}_R(\mathbf{x})$ (resp. $\text{Sym}_R(\mathbf{x})$) made of the series f which are homogeneous of degree n . Recall that a *composition of n* is a tuple of positive integers summing to n , and that the notation $(\delta_1, \dots, \delta_k) \models n$ means that $(\delta_1, \dots, \delta_k)$ is a composition of n . For $\delta = (\delta_1, \dots, \delta_k) \models n$, we denote $M_\delta \in \text{QSym}_R^n(\mathbf{x})$ the *quasisymmetric monomial function* defined by

$$M_\delta = \sum_{i_1 < i_2 < \dots < i_k} x_{i_1}^{\delta_1} x_{i_2}^{\delta_2} \dots x_{i_k}^{\delta_k},$$

where the sum is over increasing k -tuples of positive integers. It is clear that $\{M_\delta\}_{\delta \models n}$ is a basis of $\text{QSym}_R^n(\mathbf{x})$. For $f \in \text{QSym}_R(\mathbf{x})$, we denote $[M_\delta]f$ the coefficient of M_δ in the expansion of f in the basis $\{M_\delta\}_\delta$.

It is intuitively clear that $B_D(\mathbf{x}; y, z)$ is in $\text{QSym}_{\mathbb{Z}[x, y]}^{|V|}(\mathbf{x})$ (since an order-preserving modification of the colors of a coloring f of D does not change the number of ascents and descents). We now give the expansion of $B_D(\mathbf{x}; y, z)$ in the basis $\{M_\delta\}_{\delta \models |V|}$.

Proposition 8.2. *For any digraph $D = (V, A)$,*

$$B_D(\mathbf{x}; y, z) = \sum_{p=1}^{|V|} \sum_{g \in \text{Surj}(V, p)} M_{(|g^{-1}(1)|, |g^{-1}(2)|, \dots, |g^{-1}(p)|)} y^{|g_A^>|} z^{|g_A^<|}, \quad (82)$$

where $\text{Surj}(V, p)$ is the set of surjective maps from V to $[p]$.

Example 8.3. For the digraphs D , D' , D'' represented in Figure 10, one gets

$$\begin{aligned} B_D(\mathbf{x}; y, z) &= (y + z)M_{(1,1)} + M_{(2)}, \\ B_{D'}(\mathbf{x}; y, z) &= (y^2 + z^2 + 4yz)M_{(1,1,1)} + (yz + y + z)(M_{(1,2)} + M_{(2,1)}) + M_{(3)}, \\ B_{D''}(\mathbf{x}; y, z) &= 2(y^2 + z^2 + yz)M_{(1,1,1)} + (z^2 + 2y)M_{(1,2)} + (y^2 + 2z)M_{(2,1)} + M_{(3)}. \end{aligned}$$



FIGURE 10. Three (compatibly labeled) acyclic digraphs.

Proof of Proposition 8.2. The proof is almost the same as that of Theorem 3.1. Using the notation introduced in that proof, one gets

$$\sum_{f: V \rightarrow \mathbb{P}} \left(\prod_{v \in V} x_{f(v)} \right) y^{|f_A^>|} z^{|f_A^<|} = \sum_{p=1}^{|V|} \sum_{g \in \text{Surj}(V, p)} y^{|g_A^>|} z^{|g_A^<|} \sum_{f: [V] \rightarrow [q] \text{ such that } \tilde{f}=g} \left(\prod_{v \in V} x_{f(v)} \right).$$

Moreover, it is easy to see that for any $g \in \text{Surj}(V, p)$,

$$\sum_{f: [V] \rightarrow [q] \text{ such that } \tilde{f}=g} \left(\prod_{v \in V} x_{f(v)} \right) = M_{(|g^{-1}(1)|, |g^{-1}(2)|, \dots, |g^{-1}(p)|)},$$

which gives (8.2). \square

Remark 8.4. Consider the linear map φ from $\text{QSym}_R(\mathbf{x})$ to $R[q]$ (the ring of polynomials in q with coefficients in R) which for any composition $(\delta_1, \dots, \delta_k) \models n$ sends the basis element M_δ to the polynomial $\frac{q(q-1)\dots(q-k+1)}{k!}$. It is clear that for any positive integer q the map φ coincides with the evaluation map at $\mathbf{x} = 1^q$. Hence $\varphi(B_D(\mathbf{x}; y, z)) = B_D(q, y, z)$, and Proposition 8.2 is a refinement of the expansion (2) of $B_D(q, y, z)$ in the “falling factorial” basis of polynomials.

We now state the analogue of Proposition 3.2 for $B_D(\mathbf{x}; y, z)$. We denote ρ the linear map on $\text{QSym}_R(\mathbf{x})$ sending each basis element $M_{\delta_1, \delta_2, \dots, \delta_k}$ to $M_{\delta_k, \dots, \delta_2, \delta_1}$. It is clear that ρ is an involution (intuitively, ρ “reverses the order of the variables”).

Proposition 8.5. *For any digraph $D = (V, A)$, the quasisymmetric B -polynomial has the following properties.*

- (a') $\rho(B_D(\mathbf{x}; y, z)) = B_D(\mathbf{x}; z, y)$.
- (b') $B_{D-A}(\mathbf{x}; y, z) = B_D(\mathbf{x}; z, y)$.
- (c') $\sum_{f: V \rightarrow \mathbb{P}} \left(\prod_{v \in V} x_{f(v)} \right) u^{|f_A^-|} y^{|f_A^+|} z^{|f_A^-|} = u^{|A|} B_D\left(\mathbf{x}; \frac{y}{u}, \frac{z}{u}\right)$.
- (d') $\sum_{f: V \rightarrow \mathbb{P}} \left(\prod_{v \in V} x_{f(v)} \right) y^{|f_A^+|} z^{|f_A^-|} = (yz)^{|A|} B_D\left(\mathbf{x}; \frac{1}{z}, \frac{1}{y}\right)$.
- (e') The quasisymmetric B -polynomial of the digraph with a single vertex and no arcs is $M_{(1)} = \sum_{i \in \mathbb{P}} x_i$.
- (f') If $a = (u, u) \in A$ is a loop, then $B_{D \setminus a}(\mathbf{x}; y, z) = B_D(\mathbf{x}; y, z)$.
- (g') If D is the disjoint union of two digraphs D_1 and D_2 , then $B_D(\mathbf{x}; y, z) = B_{D_1}(\mathbf{x}; y, z) B_{D_2}(\mathbf{x}; y, z)$.
- (h') If D has no loops, $\deg_y(B_D(\mathbf{x}; y, y)) = |A|$.
- (j') The expansion of the polynomial $B_D(\mathbf{x}; y, z)$ in the basis $\{M_\delta y^i z^j\}_{\delta \models |V|, i, j \geq 0}$ has positive integer coefficients.
- (k') $[M_{1|V|}]B_D(\mathbf{x}; 1, 0) = \sum_{\text{bijection } f: V \rightarrow [|V|]} y^{|f_A^+|} z^{|f_A^-|}$.
- (l') For any k in $[|V|]$, $[M_{(k, |V|-k)}]B_D(\mathbf{x}; 1, 0)$ is the number of subsets of vertices U of size k such that every arc joining U to $V \setminus U$ is oriented away from U .

Proof. Property (a) follows from (82) by considering the involution on $\text{Surj}(V, p)$ which associates to any function $f \in \text{Surj}(V, p)$ the function $\bar{f}: v \mapsto p + 1 - f(v)$. The other assertions are proved with the same arguments as for Proposition 3.2. \square

Remark 8.6. Observe that

$$[M_{1, |V|-1}]B_D(\mathbf{x}; y, z) = \sum_{v \in V} y^{\text{outdeg}(v)} z^{\text{indeg}(v)},$$

where $\text{outdeg}(v)$ and $\text{indeg}(v)$ are the outdegree and indegree of v respectively. Hence, the indegree and outdegree distribution can be read off $B_D(\mathbf{x}; x, y)$. In particular, knowing $B_D(\mathbf{x}; x, y)$ (and $|A|$) is sufficient to detect whether D is a *directed tree* (an oriented tree such that every vertex except one has indegree 1). This is in contrast with $B_D(q; y, z)$ as explained in Remark 3.3.

We now show that the *profile* of acyclic digraphs can be obtained from the quasisymmetric B -polynomial. For a vertex v of an acyclic digraph D , we call *height* the maximal number of vertices on a directed path of D ending at v . We call *profile* of D , the composition $\delta = (\delta_1, \dots, \delta_k) \models |V|$, where δ_i is the number of vertices having height i . It is not hard to see that the profile of a digraph $D = (V, A)$ is the largest composition δ for the lexicographic order, such that $[M_\delta][y^{|A|}]B_D(\mathbf{x}; y, z) \neq 0$.

Next, we define the quasisymmetric version of the oriented-chromatic polynomials of digraphs. We denote

$$\chi_D^>(\mathbf{x}) = [y^A]B_D(\mathbf{x}; y, 1) = \sum_{f: V \rightarrow \mathbb{P}, f_A^>=A} \prod_{v \in V} x_{f(v)},$$

and

$$\chi_D^{\geq}(\mathbf{x}) = B_D(\mathbf{x}; 1, 0) = \sum_{f: V \rightarrow \mathbb{P}, f_A^{\geq}=A} \prod_{v \in V} x_{f(v)}.$$

Theorem 5.5 can be partially lifted to the quasisymmetric setting.

Theorem 8.7. *For any digraph $D = (V, A)$,*

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{\setminus R}^{-T}}^>(\mathbf{x}) = B_D(\mathbf{x}; 1 + y, 1 + z), \quad (83)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{\setminus R}^{-T}}^{\geq}(\mathbf{x}) = (1 + y + z)^{|A|} B_D\left(\mathbf{x}; \frac{1 + y}{1 + y + z}, \frac{1 + z}{1 + y + z}\right). \quad (84)$$

Proof. The proof is the same as for (21) and (22). \square

We will now use the theory of P -partitions in order to express $\chi_D^>(\mathbf{x})$, $\chi_D^{\geq}(\mathbf{x})$ and $B_D(\mathbf{x}, y, 1)$ in terms of the fundamental quasisymmetric function. For a subset S of $[n - 1]$, we denote $F_{n,S} \in \text{QSym}_R^n(\mathbf{x})$ the *fundamental quasisymmetric function* defined by

$$F_{n,S} = \sum_{\substack{i_1 \leq i_2 \leq \dots \leq i_n \\ i_s < i_{s+1} \text{ if } s \in S}} x_{i_1} x_{i_2} \cdots x_{i_n}.$$

It is well known that $\{F_{n,S}\}_{S \subseteq [n-1]}$ is a basis of $\text{QSym}_R^n(\mathbf{x})$ (see e.g. [32]). For $f \in \text{QSym}_R^n(\mathbf{x})$, we denote $[F_{n,S}]f$ the coefficient of $F_{n,S}$ in the expansion of f in the basis $\{F_{n,S}\}_{S \subseteq [n-1]}$.

Remark 8.8. Consider the linear map φ from $\text{QSym}_R(\mathbf{x})$ to $R[q]$ sending each basis element $F_{n,S}$ to the polynomial $\frac{(q-|S|)(q-|S|+1)\dots(q-|S|+n-1)}{n!}$. It is clear that for any positive integer q , the map φ coincides with the evaluation map at $\mathbf{x} = 1^q$. Hence, φ is the same as the linear map considered in Remark 8.4, and $\varphi(B_D(\mathbf{x}; y, z)) = B_D(q, y, z)$.

We now recall some basic results about P -partitions. The reader can refer to [33] for some background. For convenience, we will state definitions and results in terms of digraphs instead of posets.

A digraph with vertex set $[n]$ is called a *labeled digraph*. We say that D has a *compatible labeling* (resp. *antcompatible labeling*) if for all $(u, v) \in A$, $u < v$ (resp. $u > v$).

Definition 8.9. Let $D = ([n], A)$ be a labeled directed graph on n vertices. A D -*partition* is a function $f : [n] \rightarrow \mathbb{P}$ such that

- for all $(u, v) \in A$, $f(u) \leq f(v)$,
- for all $(u, v) \in A$ with $u < v$, $f(u) < f(v)$.

We denote by \mathcal{P}_D the set of D -partitions⁴, and by $\mathcal{P}_D(q)$ the subset of D -partitions f such that $f(V) \subseteq [q]$.

Example 8.10. For the labeled digraph D on the left of Figure 11, we get $\mathcal{P}_D = \{f : [3] \rightarrow \mathbb{P} \mid f(1) < f(2) \text{ and } f(1) < f(3)\}$. For the labeled digraph D' on the right of Figure 11, we get $\mathcal{P}_{D'} = \{f : [3] \rightarrow \mathbb{P} \mid f(2) \leq f(1) \text{ and } f(2) < f(3)\}$.

⁴The reader familiar with the theory of P -partitions will recognize that \mathcal{P}_D is the set of (P, Id) -partitions for the poset $P = ([n], \preceq)$, where $u \preceq v$ means that there exists a directed path from v to u .



FIGURE 11. Two labeled digraphs.

Clearly, \mathcal{P}_D is empty unless D is acyclic. Moreover, if D is an acyclic digraph with a compatible labeling, then $\mathcal{P}_D(q)$ is the set of q -colorings with only strict ascents, so that $|\mathcal{P}_D(q)| = \chi_D^>(q)$. Similarly, if D is an acyclic digraph with an anticompatible labeling, $\mathcal{P}_D(q)$ is the set of q -colorings with only weak ascents, so that $|\mathcal{P}_D(q)| = \chi_D^{\geq}(q)$.

We call *linear extension* of a digraph $D = ([n], A)$ a permutation $\sigma \in \mathfrak{S}_n$ such that for all $(u, v) \in A$, $\sigma^{-1}(u) < \sigma^{-1}(v)$ (equivalently, u appears before v in the *one-line notation* $\sigma(1)\sigma(2)\cdots\sigma(n)$ of σ). We denote $\mathcal{L}(D)$ the set of linear extensions of D . For instance, for the digraph D in Figure 11, $\mathcal{L}(D) = \{123, 132\}$. For a permutation σ of $[n]$, we denote \mathcal{P}_σ the set of functions $f : [n] \rightarrow \mathbb{P}$ such that

- for all $i \in [n-1]$, $f(\sigma(i)) \leq f(\sigma(i+1))$,
- for all $i \in [n-1]$ with $\sigma(i) < \sigma(i+1)$, $f(\sigma(i)) < f(\sigma(i+1))$.

We now state, in terms of labeled digraphs, the fundamental lemma of P -partitions (see [33, Lemma 3.15.3]).

Lemma 8.11. *For any labeled digraph $D = ([n], A)$,*

$$\mathcal{P}_D = \bigsqcup_{\sigma \in \mathcal{L}(D)} \mathcal{P}_\sigma.$$

Example 8.12. For the labeled digraph D on the left of Figure 11, we have $\mathcal{L}(D) = \{\sigma, \pi\}$ with $\sigma = 123$ and $\pi = 132$. Accordingly,

$$\mathcal{P}_D = \mathcal{P}_\sigma \uplus \mathcal{P}_\pi = \{f : [3] \rightarrow \mathbb{P} \mid f(1) < f(2) < f(3)\} \uplus \{f : [3] \rightarrow \mathbb{P} \mid f(1) < f(3) \leq f(2)\}.$$

For the labeled digraph D' on the right of Figure 11, we have $\mathcal{L}(D') = \{\sigma', \pi'\}$ with $\sigma' = 213$ and $\pi' = 231$. Accordingly,

$$\mathcal{P}_{D'} = \mathcal{P}_{\sigma'} \uplus \mathcal{P}_{\pi'} = \{f : [3] \rightarrow \mathbb{P} \mid f(2) \leq f(1) < f(3)\} \uplus \{f : [3] \rightarrow \mathbb{P} \mid f(2) < f(3) \leq f(1)\}.$$

The fundamental lemma 8.11 implies the following result about our invariants $\chi_D^>(\mathbf{x})$ and $\chi_D^{\geq}(\mathbf{x})$.

Lemma 8.13. *For any compatibly labeled acyclic digraph $D = ([n], A)$,*

$$\chi_D^>(\mathbf{x}) = \sum_{\sigma \in \mathcal{L}(D)} F_{n, \text{Asc}(\sigma)},$$

where $\text{Asc}(\sigma) = \{i \in [n-1] \mid \sigma(i) < \sigma(i+1)\}$ is the ascent set of σ . For any anticompatibly labeled acyclic digraph $D = ([n], A)$,

$$\chi_D^{\geq}(\mathbf{x}) = \sum_{\sigma \in \mathcal{L}(D)} F_{n, \text{Asc}(\sigma)}.$$

Proof. By definition, for any permutation $\sigma \in \mathfrak{S}_n$,

$$\sum_{f \in \mathcal{P}_\sigma} \prod_{v \in V} x_{f(v)} = \sum_{\substack{f(\sigma(1)) \leq f(\sigma(2)) \leq \dots \leq f(\sigma(n)) \\ \forall i \in \text{Asc}(\sigma), f(\sigma(i)) < f(\sigma(i+1))}} x_{f(\sigma(1))} x_{f(\sigma(2))} \cdots x_{f(\sigma(n))} = F_{n, \text{Asc}(\sigma)}.$$

Hence, by Lemma 8.11, for any labeled digraph D ,

$$\sum_{f \in \mathcal{P}_D} \prod_{v \in V} x_{f(v)} = \sum_{\sigma \in \mathcal{L}(D)} \sum_{f \in \mathcal{P}_\sigma} \prod_{v \in V} x_{f(v)} = \sum_{\sigma \in \mathcal{L}(D)} F_{n, \text{Asc}(\sigma)}.$$

Moreover, it is clear from the definitions, that for any compatibly labeled digraph D ,

$$\chi_D^>(\mathbf{x}) = \sum_{f \in \mathcal{P}_D} \prod_{v \in V} x_{f(v)},$$

because $\mathcal{P}_D = \{f : V \rightarrow \mathbb{P}, f_A^> = A\}$. Similarly, for any anticompatibly labeled digraph D ,

$$\sum_{f \in \mathcal{P}_D} \prod_{v \in V} x_{f(v)} = \chi_D^{\geq}(\mathbf{x}),$$

because $\mathcal{P}_D = \{f : V \rightarrow \mathbb{P}, f_A^{\geq} = A\}$. □

From Lemma 8.13, we obtain an expression for $B_D(\mathbf{x}; y, 1)$ when D is acyclic.

Theorem 8.14. *If $D = ([n], A)$ is a compatibly labeled acyclic digraph, then*

$$B_D(\mathbf{x}; y, 1) = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}(\sigma^{-1})} y^{|\sigma_A^>|}, \quad (85)$$

where $\text{Asc}(\sigma^{-1}) = \{i \in [n-1] \mid \sigma^{-1}(i) < \sigma^{-1}(i+1)\}$, and $\sigma_A^> = \{(u, v) \in A, \sigma(v) > \sigma(u)\}$. In particular (by Remark 8.8),

$$B_D(q, y, 1) = \frac{1}{n!} \sum_{\sigma \in \mathfrak{S}_n} \left(\prod_{i=1}^n (q - |\text{Asc}(\sigma^{-1})| + i - 1) \right) y^{|\sigma_A^>|}.$$

Proof. Setting $z = 0$ in (83) gives

$$B_D(\mathbf{x}; y + 1, 1) = \sum_{R \subseteq A} y^{A \setminus R} \chi_{D \setminus R}^>(\mathbf{x}).$$

Since for any set R the digraph $D \setminus R$ is compatibly labeled, Lemma 8.13 gives

$$B_D(\mathbf{x}; y + 1, 1) = \sum_{R \subseteq A} y^{|A \setminus R|} \sum_{\sigma \in \mathcal{L}(D \setminus R)} F_{n, \text{Asc}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}(\sigma)} \sum_{R \subseteq A, \text{ such that } \sigma \in \mathcal{L}(D \setminus R)} y^{|A \setminus R|}.$$

Now, observe that σ is in $\mathcal{L}(D \setminus R)$ if and only if $A \setminus R \subseteq \sigma^{-1} \stackrel{>}{A}$. Thus,

$$B_D(\mathbf{x}; y + 1, 1) = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}(\sigma)} \sum_{S \subseteq \sigma^{-1} \stackrel{>}{A}} y^{|S|} = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}(\sigma)} (y + 1)^{|\sigma^{-1} \stackrel{>}{A}|},$$

which is equivalent to (85). □

Example 8.15. Consider the digraphs D' and D'' of Figure 10. The expansion of $B_{D'}(\mathbf{x}; y, z)$ and $B_{D''}(\mathbf{x}; y, z)$ in the basis $\{M_\delta\}_\delta$ is given in Example 8.3. Using the change of basis⁵ between $\{M_\delta\}_{\delta \models n}$ and $\{F_{n,S}\}_{S \subseteq [n-1]}$ gives

$$B_{D'}(\mathbf{x}; y, z) = (y^2 + z^2 + 2yz - 2y - 2z + 1)F_{3,\{1,2\}} + (yz + y + z - 1)(F_{3,\{1\}} + F_{3,\{2\}}) + F_{3,\emptyset}, \quad (86)$$

$$B_{D''}(\mathbf{x}; y, z) = (y^2 + z^2 + 2yz - 2y - 2z + 1)F_{3,\{1,2\}} + (z^2 + 2y - 1)F_{3,\{1\}} + (y^2 + 2z - 1)F_{3,\{2\}} + F_{3,\emptyset}. \quad (87)$$

The expression for $B_{D'}(\mathbf{x}, y, 1)$ and $B_{D''}(\mathbf{x}, y, 1)$ can be checked to match those given by Theorem 8.14 as computed in Tables 2 and 3.

⁵The change of basis is given by $M_\delta = \sum_{S(\delta) \subseteq R \subseteq [n-1]} (-1)^{|R \setminus S(\delta)|} F_{n,R}$, where $S(\delta_1, \dots, \delta_k) = \left\{ \sum_{i=1}^j \delta_i, j \in [k-1] \right\}$.

σ	123	132	213	231	312	321
σ^{-1}	123	132	213	312	231	321
$ \sigma_A^> $	2	1	1	1	1	0
$\text{Asc}(\sigma^{-1})$	$\{1, 2\}$	$\{1\}$	$\{2\}$	$\{2\}$	$\{1\}$	\emptyset
$\text{Asc}_{\prec}(\sigma^{-1})$	\emptyset	$\{1\}$	$\{2\}$	\emptyset	\emptyset	\emptyset
Contribution to $B_{D'}(\mathbf{x}; y, 1)$	$y^2 F_{3, \{1, 2\}}$	$y F_{3, \{1\}}$	$y F_{3, \{2\}}$	$y F_{3, \{2\}}$	$y F_{3, \{1\}}$	$F_{3, \emptyset}$
Contribution to $\omega([z^{ A }]B_{D'}(\mathbf{x}; yz, z))$	$y^2 F_{3, \emptyset}$	$y F_{3, \{1\}}$	$y F_{3, \{2\}}$	$y F_{3, \emptyset}$	$y F_{3, \emptyset}$	$F_{3, \emptyset}$

TABLE 2. Illustration of Formulas (85) and (94) for the labeled digraph D' represented in Figure 10 (center). In the computation of Des_{\prec} , one uses the order \prec for which the unique relation is $1 \prec 3$.

σ	123	132	213	231	312	321
σ^{-1}	123	132	213	312	231	321
$ \sigma_A^> $	2	1	2	0	1	0
$\text{Asc}(\sigma^{-1})$	$\{1, 2\}$	$\{1\}$	$\{2\}$	$\{2\}$	$\{1\}$	\emptyset
$\text{Asc}_{\prec}(\sigma^{-1})$	$\{1\}$	\emptyset	\emptyset	$\{2\}$	\emptyset	\emptyset
Contribution to $B_{D'}(\mathbf{x}; y, 1)$	$y^2 F_{3, \{1, 2\}}$	$y F_{3, \{1\}}$	$y^2 F_{3, \{2\}}$	$F_{3, \{2\}}$	$y F_{3, \{1\}}$	$F_{3, \emptyset}$
Contribution to $\omega([z^{ A }]B_{D'}(\mathbf{x}; yz, z))$	$y^2 F_{3, \{1\}}$	$y F_{3, \emptyset}$	$y^2 F_{3, \emptyset}$	$F_{3, \{2\}}$	$y F_{3, \emptyset}$	$F_{3, \emptyset}$

TABLE 3. Illustration of Formulas (85) and (94) for the labeled digraph D'' represented in Figure 10 (right). In the computation of Des_{\prec} , one uses the order \prec for which the unique relation is $1 \prec 2$.

Example 8.16. Let us illustrate Theorem 8.14 for the digraphs D , D' , D'' represented in Figure 3. For the directed path on n vertices, $D = ([n], \{(i, i+1) \mid i \in [n-1]\})$, Theorem 8.14 gives

$$B_D(\mathbf{x}; y, 1) = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}(\sigma^{-1})} y^{|\text{Asc}(\sigma)|}. \quad (88)$$

For $D' = ([n], A)$ the digraph with i copies of the arc $(i, i+1)$ for all $i \in [n-1]$, Theorem 8.14 gives

$$B_{D'}(\mathbf{x}; y, 1) = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}(\sigma^{-1})} y^{\binom{n}{2} - \text{maj}(\sigma)}, \quad (89)$$

where $\text{maj}(\sigma) = \sum_{\substack{i \in [n-1] \\ \sigma(i) > \sigma(i+1)}} i$ is the *major index*. Lastly, for $D'' = ([n], \{(u, v) \mid 1 \leq u < v \leq n\})$ Theorem 8.14 gives

$$B_{D''}(\mathbf{x}; y, 1) = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}(\sigma^{-1})} y^{\binom{n}{2} - \text{inv}(\sigma)}, \quad (90)$$

where $\text{inv}(\sigma)$ is the number of pairs $(i, j) \in [n]^2$ with $i < j$ and $\sigma(i) > \sigma(j)$.

Next, we discuss certain duality results. We denote ω the linear map on $\text{QSym}_m(\mathbf{x})$ sending each basis element $F_{n, S}$ to $F_{n, [n-1] \setminus S}$. It is clear that ω is an involution. It is also well known that ω is a ring homomorphism (in other words, $\omega(fg) = \omega(f)\omega(g)$ for all $f, g \in \text{QSym}(\mathbf{x})$). Consequently, the restriction of ω to $\text{Sym}_R(\mathbf{x})$ is the well-known *duality map*, which is the unique ring homomorphism sending $e_n = F_{n, [n-1]}$ to $h_n = F_{n, \emptyset}$ for all n . We now state a key lemma which is an easy consequence of a well-known fact about the duality for quasisymmetric functions associated to labeled posets.

Lemma 8.17. *For any digraph D , $\omega(\chi_D^>(\mathbf{x})) = \mathbf{1}_{D_{\text{acyclic}}} \cdot \chi_D^>(\mathbf{x})$.*

Proof. If D is not acyclic, then clearly $\chi_D^>(\mathbf{x}) = 0$ (because a coloring cannot be strictly increasing along a directed cycle). Suppose now that D is acyclic. Up to renaming the vertices of D , we can assume that $D = ([n], A)$ is compatibly labeled. Now let $\tilde{D} = ([n], \tilde{A})$ be the (anticompatibly labeled) digraph obtained from D by relabeling the vertex i by $n+i-1$ for all $i \in [n]$. Let $\theta \in \mathfrak{S}_n$ be the permutation defined by $\theta(i) = n+1-i$ for all $i \in [n]$. It is clear that the mapping Φ defined on $\mathcal{L}(D)$ by $\phi(\sigma) = \theta \circ \sigma$ is a bijection between $\mathcal{L}(D)$ and $\mathcal{L}(\tilde{D})$. Moreover, $\text{Asc}(\sigma) = [n-1] \setminus \text{Asc}(\phi(\sigma))$, so that

$$\chi_D^>(\mathbf{x}) = \sum_{\sigma \in \mathcal{L}(D)} F_{n, \text{Asc}(\sigma)} = \sum_{\pi \in \mathcal{L}(\tilde{D})} F_{n, [n-1] \setminus \text{Asc}(\pi)} = \omega(\chi_{\tilde{D}}^>(\mathbf{x})).$$

□

As a consequence of Lemma 8.17, we get the following extensions of (37) and Theorem 6.15.

Corollary 8.18. *For any digraph D ,*

$$\sum_{\substack{R \uplus S \uplus T = A \\ D_{\setminus R}^{-T} \text{ acyclic}}} y^{|S|} z^{|T|} \chi_{D_{\setminus R}^{-T}}^>(\mathbf{x}) = \omega(B_D(\mathbf{x}, 1+y, 1+z)). \quad (91)$$

Proof. Apply the involution ω to both sides of (83) and use Lemma 8.17. □

Corollary 8.19. *If $D = (V, A)$ is an acyclic digraph, then*

$$\omega(B_D(\mathbf{x}; y, 1)) = \rho\left(y^{|A|} B_D(\mathbf{x}; 1/y, 1)\right). \quad (92)$$

If the graph underlying D is a forest, then

$$\omega(B_D(\mathbf{x}; y, z)) = (y+z-1)^{|A|} B_D\left(\mathbf{x}; \frac{y}{y+z-1}, \frac{z}{y+z-1}\right). \quad (93)$$

Proof. If D is acyclic, then all its subgraphs are acyclic. Hence, setting $z = 0$ in (91) gives

$$\sum_{R \subseteq A} y^{|A \setminus R|} \chi_{D_{\setminus R}}^>(\mathbf{x}) = \omega(B_D(\mathbf{x}; y+1, 1)).$$

Comparing this to the specialization $z = 0$ of (84) gives $\omega(B_D(\mathbf{x}; y+1, 1)) = (y+1)^{|A|} B_D(\mathbf{x}; 1, 1/(y+1))$ which is equivalent to (92).

If the graph underlying D is a forest, then any digraph obtained from D by deleting or reorienting arcs is acyclic. Hence, (91) gives

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{\setminus R}^{-T}}^>(\mathbf{x}) = \omega(B_D(\mathbf{x}; 1+y, 1+z)).$$

Comparing this to (84) gives $\omega(B_D(\mathbf{x}; 1+y, 1+z)) = (1+y+z)^{|A|} B_D\left(\mathbf{x}; \frac{1+y}{1+y+z}, \frac{1+z}{1+y+z}\right)$, which is equivalent to (93). □

Remark 8.20. In [24] a surprising identity is proved for $\mathcal{X}_D(\mathbf{x}; y) = [z^{|A|}] B_D(\mathbf{x}; yz, z)$ for particular digraphs. We state this result with our notation for the reader's convenience. Let $D = ([n], A)$ be a compatibly labeled acyclic digraph without double arcs. Suppose that there exists a partial order \prec on V such that there is an arc $(u, v) \in A$ if and only if u, v are incomparable for \prec . Then, [24, Theorem 3.1] states that

$$\omega([z^{|A|}] B_D(\mathbf{x}; yz, z)) = \sum_{\sigma \in \mathfrak{S}_n} F_{n, \text{Asc}_{\prec}(\sigma^{-1})} y^{|\sigma_A^>|}, \quad (94)$$

where $\text{Asc}_{\prec}(\pi) = \{i \in [n-1] \mid \pi(i) \prec \pi(i+1)\}$. For instance, for the digraphs D' and D'' of Figure 10, the expansion of $B_{D'}(\mathbf{x}; y, z)$ and $B_{D''}(\mathbf{x}; y, z)$ in the fundamental basis is given in (86)

and (87). The expression for $[z^{|A|}]B_{D'}(\mathbf{x}, yz, z)$ and $[z^{|A|}]B_{D''}(\mathbf{x}, yz, z)$ can be checked to match those given by (94) as computed in Tables 2 and 3.

9. A FAMILY OF INVARIANTS GENERALIZING THE B -POLYNOMIAL

In this section we define a family of digraph invariants generalizing the B -polynomial.

Theorem 9.1. *Let m be a positive integer. For any digraph $D = (V, A)$ there exists a unique polynomial in $2m + 1$ variables $B_D^{(m)}(q; y_1, \dots, y_m; z_1, \dots, z_m)$ such that, for every non-negative integer p ,*

$$B_D^{(m)}(mp + 1; y_1, \dots, y_m; z_1, \dots, z_m) = \sum_{f: V \rightarrow [mp+1]} \prod_{k=1}^m y_k^{|f_A^k|} z_k^{|f_A^{-k}|}, \quad (95)$$

where f_A^k (resp. f_A^{-k}) is the set of arcs $(u, v) \in A$ such that $f(v) - f(u) \in [(k-1)p + 1 .. kp]$ (resp. $f(u) - f(v) \in [(k-1)p + 1 .. kp]$).

For instance, $B_D^{(1)}(q; y_1; z_1)$ is equal to the B -polynomial $B_D(q, y_1, z_1)$. The proof of Theorem 9.1 uses the following classical extension of Ehrhart's Theory to rational polytopes.

Theorem 9.2. *Let $\Pi \subset \mathbb{R}^n$ be a polytope and let m be an integer. Suppose that all the coordinates of the vertices of Π are in $\frac{1}{m}\mathbb{Z}$. Then, there exists polynomials E_0, E_1, \dots, E_{m-1} such that for all non-negative integers p and all $r \in [0..m-1]$,*

$$|(mp + r)\Pi \cap \mathbb{Z}^n| = E_r(mp + r).$$

Moreover, for positive integer p and all $r \in [0..m-1]$,

$$|(mp - r)\Pi^\bullet \cap \mathbb{Z}^n| = (-1)^{\dim(\Pi)} E_r(-mp + r), \quad (96)$$

where Π^\bullet is the relative interior of Π (that is, the interior of Π in the smallest affine subspace containing Π), and $\dim(\Pi)$ is the dimension of the smallest affine subspace containing Π .

Example 9.3. The 2-dimensional polytope $\Pi = [0, 1/3]^2$ has vertex-coordinates in $\frac{1}{m}\mathbb{Z}$ for $m = 3$. The polynomials are $E_i(q) = (\frac{q-i}{3} + 1)^2$ for all $i \in \{0, 1, 2\}$. Since $\Pi^\bullet = (0, 1/3)^2$, we get $|\Pi^\bullet \cap \mathbb{Z}^2| = 0 = E_2(-1)$, which is in accordance with (96).

Proof of Theorem 9.1. The uniqueness part is obvious, so we focus on the proof of the existence of $B_D^{(m)}$. For a function $g : A \rightarrow [m..m]$, we denote $\chi_D^g(mp + 1)$ the number of $(mp + 1)$ -colorings $f : V \rightarrow [mp + 1]$ such that for all $k \in [m]$, $f_A^k = g^{-1}(k)$ and $f_A^{-k} = g^{-1}(-k)$. We clearly have

$$\sum_{f: V \rightarrow [mp+1]} \prod_{k=1}^m y_k^{|f_A^k|} z_k^{|f_A^{-k}|} = \sum_{g: A \rightarrow [-m..m]} \chi_D^g(mp + 1) \prod_{k=1}^m y_k^{|g^{-1}(k)|} z_k^{|g^{-1}(-k)|}.$$

Thus, it suffices to prove that for any function $g : A \rightarrow [-m..m]$, there exists a polynomial P such that $\chi_D^g(mp + 1) = P(mp + 1)$ for all non-negative integer p .

To a function $g : A \rightarrow [-m..m]$, we associate a region Δ_D^g of \mathbb{R}^V defined at follows:

$$\Delta_D^g = \{(x_v)_{v \in V} \in \mathbb{R}^V \mid \forall v \in V, x_v \in (0, 1], \text{ and } \forall a = (u, v) \in A, x_v - x_u \in I_{g(a)}\},$$

where $I_0 = \{0\}$ and for all $k \in [m]$, $I_k = ((k-1)/m, k/m)$ and $I_{-k} = -I_k = (-k/m, -(k-1)/m)$. Clearly, for all non-negative integers p ,

$$\chi_D^g(mp + 1) = |(mp + 1)\Delta_D^g \cap \mathbb{Z}^V|,$$

because for all $k \in [m]$, $(mp + 1)I_k \cap \mathbb{Z} = [(k-1)p + 1 .. kp]$.

For a subset of vertices $U \subseteq V$, we denote

$$\Delta_{D,U}^g = \{(x_v)_{v \in V} \in \mathbb{R}^V \mid \forall v \in U, x_v = 1, \forall v \in V \setminus U, x_v \in (0, 1), \text{ and } \forall a = (u, v) \in A, x_v - x_u \in I_{g(a)}\},$$

and $\chi_{D,U}^g(mp+1) = \left| (mp+1)\Delta_{D,U}^g \cap \mathbb{Z}^V \right|$. With this notation, one gets $\Delta_D^g = \biguplus_{U \subseteq V} \Delta_{D,U}^g$, and

$\chi_D^g(mp+1) = \sum_{U \subseteq V} \chi_{D,U}^g(mp+1)$. It remains to show that there exists a polynomial $P_{g,U}$ such

that for all non-negative integers p , $\Delta_{D,U}^g(mp+1) = P_{g,U}(mp+1)$. If $\Delta_{D,U}^g = \emptyset$, then we can take $P_{g,U} = 0$. Hence, we can now assume that $\Delta_{D,U}^g$ is non-empty. Since $\Delta_{D,U}^g$ is non-empty, it is the relative interior of the polytope

$$\overline{\Delta}_{D,U}^g = \{(x_v)_{v \in V} \in \mathbb{R}^V \mid \forall v \in U, x_v = 1, \forall v \in V \setminus U, x_v \in [0, 1], \text{ and } \forall a = (u, v) \in A, x_v - x_u \in \overline{I}_{g(a)}\},$$

where $\overline{I}_0 = \{0\}$ and for all $k \in [m]$, $\overline{I}_k = [(k-1)/m, k/m]$ and $\overline{I}_{-k} = [-k/m, -(k-1)/m]$.

We now want to conclude using Ehrhart theory, and need to show that the vertices of $\overline{\Delta}_{D,U}^g$ are in $\frac{1}{m}\mathbb{Z}^V$. Any vertex $(x_v)_{v \in V}$ of $\overline{\Delta}_{D,U}^g$ is the intersection of $|V|$ hyperplanes $h_1, \dots, h_{|V|}$ of the form $H_{(u,v),c} := \{x_v - x_u = c\}$, or $H_{v,c} := \{x_v = c\}$ with $c \in \frac{1}{m}\mathbb{Z}$. It is well-known that the incidence matrix of any graph is totally unimodular, so the determinant of the system of linear equations given by $h_1, \dots, h_{|V|}$ is either 1 or -1 , and its solution $(x_v)_{v \in V}$ is in $\frac{1}{m}\mathbb{Z}^V$. Consequently, by Theorem 9.2 there is a polynomial E_{m-1} such that for any non-negative integer p ,

$$\chi_{D,U}^g(mp+1) = \left| (mp+1)\overline{\Delta}_{D,U}^g \cap \mathbb{Z}^V \right| = (-1)^{\dim(\overline{\Delta}_{D,U}^g)} E_{m-1}(-mp-1).$$

So we can take $P_{g,U}(q) = (-1)^{\dim(\overline{\Delta}_{D,U}^g)} E_{m-1}(-q)$, which concludes the proof. \square

Observe that for any positive m , the invariant $B_D^{(m)}$ refines the B -polynomial, since $B^{(m)}(q; y, \dots, y; z, \dots, z) = B_D(q, y, z)$. In particular, the polynomial $B_D^{(m)}$ can be specialized to the Potts polynomial of the underlying graph \underline{D} :

$$B_D^{(m)}(q; y, \dots, y; y, \dots, y) = B_D(q, y, y) = P_{\underline{D}}(q, y). \quad (97)$$

We will now focus on trivariate specializations of $B_D^{(m)}$ which generalize to digraphs the Potts polynomial of graphs.

Definition 9.4. For a binary word $w = w_1, w_2, \dots, w_m \in \{-1, 1\}^m$, we denote

$$B_D^w(q, y, z) = B_D^{(m)}(q, y_1, \dots, y_m; z_1, \dots, z_m) \Big|_{\begin{array}{ll} y_k = y \text{ and } z_k = z & \text{if } w_k = 1 \\ y_k = z \text{ and } z_k = y & \text{if } w_k = -1 \end{array}}.$$

Equivalently, $B_D^w(q, y, z)$ is the unique polynomial such that for all positive integers p

$$B_D^w(mp+1, y, z) = \sum_{f: V \rightarrow [mp+1]} y^{|f_A^{>w}|} z^{|f_A^{<w}|},$$

where $f_A^{>w}$ (resp $f_A^{<w}$) is the set of arcs $(u, v) \in A$ such that $\frac{f(v)-f(u)}{p}$ (resp. $\frac{f(u)-f(v)}{p}$) is in the set

$$I(w) := \biguplus_{k=1}^m w_k \cdot ((k-1), k], \quad (98)$$

where $-((k-1), k]$ means $[-k, -(k-1))$.

For instance, with this notation, we have $B_D^1(q, y, z) = B_D^{(1)}(q, y, z) = B_D(q, y, z)$. Note that the invariants $B_D^w(q, y, z)$ are not all distinct, since for instance $B_D^{1,1}(q, y, z) = B_D^1(q, y, z) = B_D^{-1}(q, y, z)$. We now claim that the relations (8), (10), and (11) between the B -polynomial and the Tutte polynomial hold more generally for B^w -polynomials.

Proposition 9.5. *Let $w = w_1 w_2 \cdots w_m \in \{-1, 1\}^m$. For any graph G ,*

$$B_{\vec{G}}^w(q, y, z) = P_G(q, yz), \quad (99)$$

and

$$\frac{1}{2^{|E|}} \sum_{\vec{G} \in \text{Orient}(G)} B_{\vec{G}}^w(q, y, z) = P_G\left(q, \frac{y+z}{2}\right), \quad (100)$$

where the sum is over all digraphs \vec{G} obtained by orienting G . Moreover, for any digraph D ,

$$B_D^w(q, y, y) = P_{\underline{D}}(q, y), \quad (101)$$

where \underline{D} is the digraph underlying D .

Proof. The proofs of (8) and (10) extend almost verbatim to prove (99) and (100). For the proof of (99) we use the fact that $y_k z_k = yz$ for all $k \in [m]$, whereas for the proof of (100), we use the fact that $y_k + z_k = y + z$ for all $k \in [m]$. Finally, (101) is just a reformulation of (97). \square

Equation (99) shows that although the invariants $B_D^w(q, y, z)$ are not all equal, they all coincide when considering only undirected graphs, and are legitimate generalizations of the Potts polynomial. We now explore the other properties of the B -polynomial which extend to the B^w -polynomials. First note that the Properties (a)-(h) of Proposition 3.2 extend to B^w , up to changing (c) and (d) into

$$\begin{aligned} \text{(c')} \quad & \sum_{f: V \rightarrow [mp+1]} x^{|f_A^-|} y^{|f_A^{>w}|} z^{|f_A^{<w}|} = x^{|A|} B_D^w\left(q, \frac{y}{x}, \frac{z}{x}\right), \\ \text{(d')} \quad & \sum_{f: V \rightarrow [mp+1]} y^{|f_A^{>w}|} z^{|f_A^{<w}|} = (yz)^{|A|} B_D^w\left(q, \frac{1}{y}, \frac{1}{z}\right), \end{aligned}$$

where $f_A^{>w}$ (resp. $f_A^{<w}$) is the set of arcs $(u, v) \in A$ such that $\frac{f(v)-f(u)}{p}$ (resp. $\frac{f(v)-f(u)}{p}$) is in the set $\{0\} \cup I(w)$. Also the recurrence relations of the B -polynomial given in Lemmas 4.1 and 4.2 extend verbatim to the B^w -polynomials.

We now define the analogues of the strict and weak-chromatic polynomials, and generalize Theorem 5.5 to the invariants B^w .

Definition 9.6. Let $D = (V, A)$ be a digraph and let $w = w_1 w_2 \cdots w_m \in \{-1, 1\}^m$. The *w-strict-chromatic polynomial* of D is the unique polynomial $\chi_D^{>w}(q)$ such that for all non-negative integers p ,

$$\chi_D^{>w}(mp+1) = \left| \left\{ f : V \rightarrow [mp+1] \mid \forall (u, v) \in A, \frac{f(v)-f(u)}{p} \in I(w) \right\} \right|,$$

where $I(w)$ is defined by (98). The *w-weak-chromatic polynomial* of D is the unique polynomial $\chi_D^{\geq w}(q)$ such that for all non-negative integers p ,

$$\chi_D^{\geq w}(mp+1) = \left| \left\{ f : V \rightarrow [mp+1] \mid \forall (u, v) \in A, \frac{f(v)-f(u)}{p} \in \{0\} \cup I(w) \right\} \right|.$$

Note that $\chi_D^{>w}(q) = [y^{|A|}] B_D^w(mp+1, y, 1)$ and $\chi_D^{\geq w}(q) = B_D^w(q, 0, 1)$. Moreover, $\chi_D^{>1} = \chi_D^{>}$ and $\chi_D^{\geq 1} = \chi_D^{\geq}$. We now state the generalization of Theorem 5.5.

Theorem 9.7. *Let $D = (V, A)$ be a digraph and let $w = w_1 w_2 \cdots w_m \in \{-1, 1\}^m$. Then,*

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{\setminus R}^{-T}}^{>w}(q) = B_D^w(q, 1+y, 1+z), \quad (102)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{\setminus R}^{-T}}^{\geq w}(q) = (1+y+z)^{|A|} B_D^w\left(q, \frac{1+y}{1+y+z}, \frac{1+z}{1+y+z}\right), \quad (103)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{/R}^{-T}}^{>w}(q) = B_D^w(q, y, z), \quad (104)$$

$$\sum_{R \uplus S \uplus T = A} y^{|S|} z^{|T|} \chi_{D_{/R}^{-T}}^{\geq w}(q) = (1+y+z)^{|A|} B_D^w\left(q, \frac{y}{1+y+z}, \frac{z}{1+y+z}\right). \quad (105)$$

Proof. Again, the proof of Theorem 5.5 extends without difficulty. For all non-negative integers p ,

$$B_D^w(mp+1, y, z) = \sum_{f: V \rightarrow [mp+1]} \prod_{(u,v) \in A} \theta_w\left(\frac{f(v) - f(u)}{p}\right)$$

where for all $r \in [-m, m]$,

$$\theta_w(r) = \begin{cases} y & \text{if } r \in I(w), \\ z & \text{if } -r \in I(w), \\ 1 & \text{if } r = 0. \end{cases}$$

Then for all $r \in [-m, m]$,

$$\theta_w(r) = x_1 + x_2 \mathbb{1}_{r \in I(w)} + x_3 \mathbb{1}_{-r \in I(w)}, \text{ for } (x_1, x_2, x_3) = (1, y-1, z-1), \quad (106)$$

$$\theta_w(r) = x_1 + x_2 \mathbb{1}_{r \in \{0\} \cup I(w)} + x_3 \mathbb{1}_{-r \in \{0\} \cup I(w)}, \text{ for } (x_1, x_2, x_3) = (y+z-1, 1-z, 1-y), \quad (107)$$

$$\theta_w(r) = x_1 \mathbb{1}_{r=0} + x_2 \mathbb{1}_{r \in I(w)} + x_3 \mathbb{1}_{-r \in I(w)}, \text{ for } (x_1, x_2, x_3) = (1, y, z), \quad (108)$$

$$\theta_w(r) = x_1 \mathbb{1}_{r=0} + x_2 \mathbb{1}_{r \in \{0\} \cup I(w)} + x_3 \mathbb{1}_{-r \in \{0\} \cup I(w)}, \text{ for } (x_1, x_2, x_3) = (1-y-z, y, z). \quad (109)$$

Then, (106-109) imply (102-105) as in the proof of Theorem 5.5. \square

It should be mentioned that other properties of the B -polynomial, such as those established in Sections 6 and 7 do not extend to the B^w -polynomials for arbitrary w .

We now focus on a subfamily of the B^w -polynomials which enjoy additional properties. We say that a word $w = w_1 \dots w_m \in \{-1, 1\}^m$ is *antipalindromic* if for all $k \in [m]$, $w_{m+1-k} = -w_k$. We will show that when w is antipalindromic, the invariant B_D^w is essentially determined by the oriented matroid underlying D .

We start with the key observation. Let $D = (V, A)$ be a digraph, and let f be a q -coloring of D . We denote $\bar{f}: A \rightarrow \mathbb{Z}/q\mathbb{Z}$ the function defined by $\bar{f}(u, v) = f(v) - f(u) \bmod q$, and we call \bar{f} a q -coflow of D .

Lemma 9.8. *Each q -coflow of D corresponds to $q^{c(D)}$ q -colorings. Moreover, if w is antipalindromic, then for all non-negative integers p*

$$B_D^w(mp+1, y, z) = (mp+1)^{c(D)} \sum_{\bar{f}: A \rightarrow \mathbb{Z}/(mp+1)\mathbb{Z}, (mp+1)\text{-coflow}} y^{\bar{f}_A^{>w}} z^{\bar{f}_A^{<w}},$$

where $\bar{f}_A^{>w}$ (resp. $\bar{f}_A^{<w}$) is the set of arcs $a \in A$ such that $\bar{f}(a) \equiv (k-1)p + r \bmod (mp+1)$, with $r \in [p]$ and with $k \in [m]$ such that $w_k = 1$ (resp. $w_k = -1$).

Proof. First, it is clear that two q -colorings f, g satisfy $\bar{f} = \bar{g}$ if and only if for each component D' of D , there is a constant c such that for every vertex v of D' , $f(v) \equiv g(v) + c \bmod q$. So for each q -coloring f , there are $q^{c(D)}$ q -colorings g such that $\bar{g} = \bar{f}$.

We now suppose that w is antipalyndromic. It suffices to prove that any $(mp+1)$ -coloring f satisfies $f_A^{>w} = \bar{f}_A^{>w}$ and $f_A^{<w} = \bar{f}_A^{<w}$. By definition, $(u, v) \in f_A^{>w}$ if and only if there exists $k \in [m]$ such that either $w_k = 1$ and $(k-1)p+1 \leq f(v) - f(u) \leq kp$, or $w_k = -1$ and $-kp \leq f(v) - f(u) \leq -(k-1)p-1$. Since w is antipalyndromic, the second case can be rewritten as $w_{m+1-k} = 1$ and $(m+1-k)p - (mp+1) \leq f(v) - f(u) \leq (m+1-k)p - (mp+1)$. Thus, $(u, v) \in f_A^{>w}$ if and only if $(u, v) \in \bar{f}_A^{>w}$. Similarly, $(u, v) \in f_A^{<w}$ if and only if $(u, v) \in \bar{f}_A^{<w}$. \square

Recall that each digraph D has an underlying *oriented matroid* M_D , which is the oriented matroid whose cycles are given by the simple cycles of D . We refer the reader to [9] for definitions about oriented matroids, but we will not use any result from oriented matroid theory.

Corollary 9.9. *If $w \in \{-1, 1\}^m$ is antipalyndromic, then for any digraphs D , the polynomial $B_D^w(q, y, z)$ is divisible by $q^{c(D)}$. Moreover, the polynomial invariant $q^{-c(D)} B_D^w(q, y, z)$ only depends on the oriented matroid M_D underlying D (that is, $M_D = M_{D'}$ implies $q^{-c(D)} B_D^w(q, y, z) = q^{-c(D')} B_{D'}^w(q, y, z)$).*

Proof. Let D be a connected digraph. Let a, b be non-negative integers, and let $P(q) := [y^a z^b] B_D^w(q, y, z)$. We know that $P(q)$ is a polynomial in q and we want to show that it is divisible by q . By Lemma 9.8, we know that for any non-negative integer p , the value $P(mp+1)$ is an integer divisible by $mp+1$. Since the polynomial $Q(p) = P(mp+1)$ is integer valued, $Q(p)$ has rational coefficients, so that $P(q)$ also has rational coefficients. Hence, there is a rational number r and a polynomial $R(q)$ with rational coefficients such that $\frac{1}{q}P(q) = r/q + R(q)$. Let d be the least common multiple of the denominators of the coefficients of R . Then for all non-negative integers p , $r/(mp+1) \in \frac{1}{d}\mathbb{Z}$. Taking p large enough shows that $r = 0$. Hence, $\frac{1}{q}P(q)$ is a polynomial. This proves that q divides $B_D^w(q, y, z)$. Since the invariant B^w is multiplicative over connected components, this implies that for any digraph $q^{c(D)}$ divides $B_D^w(q, y, z)$.

Next, we show that the polynomial $q^{-c(D)} B_D^w$ only depends on the oriented matroid M_D underlying D . By Lemma 9.8 we only need to show that for any positive integer q , the set of q -coflows only depends on M_D . Moreover, it is easy to see that a function $g : A \rightarrow \mathbb{Z}/q\mathbb{Z}$ is a q -coflow if and only if along any simple cycle C of D ,

$$\sum_{a \in C^-} g(a) \equiv \sum_{a \in C^+} g(a),$$

where C^- and C^+ are the sets of arc in one direction and the other direction along D . This characterization only depends on the simple cycles of D , hence only on M_D . \square

Remark 9.10. By contrast to the case of antipalyndromic words w , it is clear that the invariant B_D is *not* determined by the underlying oriented matroid M_D and the number of components. Indeed, by Corollary 5.7, the invariant B_D detects the length of directed paths which cannot be detected from M_D (for instance every oriented tree has the same underlying oriented matroid).

In the companion paper [3] we will investigate in more detail the invariant $q^{-c(D)} B_D^w$ corresponding to the simplest antipalyndromic word $w = 1, -1$, for digraphs and more generally for regular oriented matroids. The invariant $q^{-c(D)} B_D^{1, -1}$ is denoted A_D in that paper.

10. CONCLUDING REMARKS

We have shown that the B -polynomial is a natural generalization of the Potts polynomial (or equivalently, Tutte Polynomial) to digraphs. We have explored several aspects of this invariant. However, this leaves many more open questions, besides the two combinatorial puzzles already mentioned in Questions 6.17 and 7.10.

One subject we did not attempt to cover is the computational complexity of determining the B -polynomial of a given digraph D . Given that $B_D(q, y, z)$ is a polynomial in q of degree $|V|$, it is uniquely determined by the values $\{B_D(q, y, z)\}$ for all $q \in [|V|]$, and the fact that $B_D(0, y, z) = 0$. Hence, using a naive approach gives a way of determining $B_D(q, y, z)$ in $O(|A| \cdot |V|^{|V|})$ operations. Using (2) actually gives a slightly better complexity of $\ln(-2)^{|V|}|V|!$ up to a subexponential factor (and actually (82) gives the same complexity for computing the quasisymmetric Tutte polynomial). We do not know if a more efficient method exists.

Question 10.1. What is the complexity of computing $B_D(q, y, z)$ in terms of $|V|$? What are the specializations of $B_D(q, y, z)$ which are computable in polynomial time?

In Section 7, we gave an interpretation of $T_D^{(1)}(2, 0)$, $T_D^{(2)}(2, 0)$, and $T_D^{(2)}(0, 2)$ as counting some classes of orientations of the mixed graph D . In view of the known results for the Tutte polynomials, we ask the following question.

Question 10.2. For $i \in \{1, 2\}$, is there an interpretations for the evaluations $T_D^{(i)}(1, 1)$, $T_D^{(i)}(0, 1)$, $T_D^{(i)}(1, 0)$, $T_D^{(i)}(2, 1)$, and $T_D^{(i)}(1, 2)$ as counting some classes of orientations of a mixed graph D ?

Recall from Proposition 3.2 that $B_D(q, y, z)$ is symmetric in y and z . Therefore $B_D(q, y, z)$ is a polynomial in q , yz and $y + z$. Moreover, when D corresponds to a graph (that is, $D = \overleftrightarrow{G}$ for some G), then Proposition 3.5 shows that $B_D(q, y, z)$ is actually a polynomial in q and yz only. It would be interesting to know if the following converse is true.

Question 10.3. Is it true that $B_D(q, y, z)$ is a function of q and yz if and only if $D = \overleftrightarrow{G}$ for a graph G ?

For planar digraphs, we have established a duality relation (50) for the specialization $B_D(-1, y, z)$ of the B -polynomial, while the classical duality relation for the Tutte polynomial gives a duality relation (53) for the specialization $B_D(q, y, y)$. We have not found a common generalization of these two relations in general.

Question 10.4. Is there an infinite class of planar digraphs for which there exists a common generalization of the two duality relations (50) and (53)?

Let D a digraph. Recall from Remark 3.8 that the number of spanning trees of the underlying graph \underline{D} can be obtained from $B_D(q, y, z)$. In contrast, Remark 3.3 shows that the number of *directed spanning trees* of D (spanning trees oriented in such a way that every vertex except one has indegree 1) cannot be obtained from D .

Question 10.5. Is it possible to obtain from $B_D(q, y, z)$ the number of *alternating spanning trees* of D , that is, spanning trees T of D such that every vertex of T is either a source or a sink of T ? Is it possible to obtain from $B_D(\mathbf{x}; y, z)$ either the number of *directed spanning trees* of D or the number of *alternating spanning trees* of D ?

Lastly, one can wonder which classes of digraphs are distinguished by $B_D(q, y, z)$, and by $B_D(\mathbf{x}; y, z)$. We say that a digraph invariant *distinguishes between* a class C of digraphs if the digraphs in C can be determined from the value of the invariant (equivalently, there does not exist non-isomorphic digraphs in C with the same value of the invariant). It is clear that $B_D(\mathbf{x}; y, z)$ does not distinguish between all digraphs, and not even between all the digraphs corresponding to graphs. Indeed, by (78) for all graphs G , $B_{\overleftrightarrow{G}}(\mathbf{x}; y, z)$ is equivalent to Tutte symmetric function $S_G(\mathbf{x}; y)$, and this invariant does not distinguish between all graphs [31]. However, in [30] Stanley raised the question of whether the chromatic symmetric function can distinguish between all trees. This question has generated a lot of interest, but is still open [2, 20, 1, 21]. Let us also mention the article [22], in which some necessary conditions are given for two acyclic digraphs to have the

same weak-chromatic polynomial (and more generally, conditions for labeled posets to have the same quasisymmetric functions). Similar questions are natural here.

Question 10.6. Does the quasisymmetric B -polynomial distinguish between

- (i) all *oriented trees* (digraph whose underlying graph is a tree)?
- (ii) all *directed trees* (oriented trees such that every vertex except one has indegree 1)?
- (iii) all *alternating trees* (oriented trees such that every vertex is either a source or a sink)?
- (iv) all *alternating digraphs* (digraphs such that every vertex is either a source or a sink)?

Answering either cases (i), (ii) or (iii) of Question 10.6 would hopefully shed some light on Stanley's original question. Questions (ii) and (iii) seems intuitively more tractable than Stanley's original question. For instance, it is clear from Remark 8.6 that the quasisymmetric B -polynomial distinguishes between the directed caterpillars whose root-vertex (vertex of indegree 0) is at an extremity of the spine (because such caterpillars are distinguished by their profile). However, to our shame, we do not know if a positive answer to Stanley's original question would give a positive answer to cases (ii) or (iii). Indeed we do not know the answer to the following seemingly easier questions.

Question 10.7. (i) Let G be any fixed graph. Does the quasisymmetric B -polynomial distinguish between all the non-isomorphic digraphs obtained by orienting G ?

- (ii) Let T be any fixed tree. Does the quasisymmetric B -polynomial distinguish between all the non-isomorphic oriented trees obtained by orienting T ?
- (iii) Let T be any fixed tree. Does the quasisymmetric B -polynomial distinguish between all the non-isomorphic directed trees obtained by orienting T ?
- (iv) Let T be any fixed tree. Does the quasisymmetric B -polynomial distinguish between all the non-isomorphic alternating trees obtained by orienting T ?

Lastly, let us mention that the motivation for the case (iv) of Question 10.6 is that the class of alternating digraphs identifies with the class of *hypergraphs*. By *hypergraph* we mean a bipartite graph $G = (V, E)$ with a prescribed bipartition of the vertex set $V = V_1 \uplus V_2$. The identification with alternating digraphs is simply obtained by orienting every edge from V_1 to V_2 . Because of this correspondence, it would be interesting to study what properties of alternating digraphs can be obtained from $B_D(q, y, z)$, and from $B_D(\mathbf{x}; y, z)$.

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